Surfing on the ocean of numbers
- a Few Smarandache Notions and Similar Topics

Henry Ibstedt

Erhus University Press
Vail
1997
Surfing on the ocean of numbers
-a Few Smarandache Notions and Similar Topics

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The cover picture refers to the last article in this book. It illustrates cumulative statistics on the occurrence of square free integers with 1, 2, 3, etc. prime factors for successive intervals of integers. A detailed discussion is given in the article.
To the Memory of

my Son

Carl-Magnus
Preface

*Surfing on the Ocean of Numbers* - why this title? Because this little book does not attempt to give theorems and rigorous proofs in the theory of numbers. Instead it will attempt to throw light on some properties of numbers, notably integers, through a study of the behaviour of large numbers of integers in order to draw some reasonably certain conclusions or support already made conjectures. But no matter how far we extend our search or increase our samples in these studies we are in fact, in spite of more and more powerful technologies, merely skimming the surface of the immense sea of numbers. Hence the title.

Most books in Mathematics are used as reference books. I still consult my first Number Theory book which I bought in 1949 - *Elementary Number Theory* by Uspensky and Heaslet. It was when I was young and enthusiastic and dreamt about becoming a Mathematician. I am still enthusiastic but I became a Physicist instead. However, I stayed on the theoretical side and avoided to have to much to do with things that can break. But even so experiments are a major source of knowledge and maybe this book shows a little of a Physicist’s approach to Mathematics. Most results are presented or supported by tables and graphs. All calculations have been carried out on a Pentium 100 MHz laptop using Ubasic as a programming language. Finally, the author has tried to make a book which should be easy and pleasant to read.

A word about the beauty of Mathematics and Number Theory in particular. The crystallized truth of a theorem, where a whole spectrum of mathematical thoughts come together to form an entity, is like a painting where designs and colours merge into a work of art. But sometimes it is not the finished result which is the most interesting - it could be the unsolved problem itself. Why? Maybe it is the challenge of getting somewhere with it or the hours and days of thinking and trying that occupy the mind in a positive sense different from the problems of our time. It all brings peace to the mind - it’s like walking in the silence of the forest enjoying the trees, the sun and the blue sky, and should it happen that all the bits and pieces suddenly fall into place to give a solution then it is the most sublime experience for the human mind - eureka! But then the interest in the problem fades away unless solving the problem created new ones - and that is almost always the case.
Most topics in this book have been selected from Only Problems, Not Solutions by F. Smarandache. Others have been suggested by Dr. R. Muller of Erhus University Press. A few problems which the author has found interesting originate from the Numbers Count Column of Personal Computer World. This journal has had great importance for the author as a source of recreational Mathematics and I take this opportunity to thank the Editor of this column Mike Mudge for all correspondence and encouragement he gave me in the past.

Illustrations, graphics, layout and final editing up to camera ready form has been done by the author. Tables have been created by direct transfer from computer files established at the time of computation to the manuscript so as to avoid typing errors.

This book has come into being thanks to R. Muller at Erhus University Press who has never failed an opportunity to give his support and encouragement. Rapid e-mail exchange between him in the USA and me in France has greatly facilitated our work. I also thank Dr. Muller’s colleagues for their help. Many thanks are also due to my son Michael Ibstedt for his help and advice concerning computer equipment and software.

Last but not least my warm thanks to my dear wife Anne-Marie for her encouragement and endless patience with a husband who does not always listen because his mind is somewhere else.

February 1997
Henry Ibstedt

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The Equation y = 2·x_1·x_2·…·x_k + 1

9
13
19
24
27
30
39
45
46
52
59
68
70
Chapter I

On Prime Numbers

This chapter deals with some computational observations on prime numbers and their distribution. These computations only skim the surface of the ocean of integers but they give an idea of the general behavior of primes and often support some of the many conjectures that are made concerning primes. Computer programs have been written in Ubasic with extensive use of some of the built in functions of this language. In some cases the use of these functions will be illustrated with a few lines of program code. Most results are given in tabular and/or graphical form.

1. On the Sequence $a \cdot p_n + b$

In his book *Only Problems, Not Solutions*¹ F. Smarandache asks the following question:

If $(a, b)=1$, how many primes does the progression $a \cdot p_n + b$, where $p_n$ is prime and $n \in \{1, 2, \ldots\}$, contain?

Already for $a=1$ and $b=2$ we run into the classical unsolved problem "Are there infinitely many twin primes?". The answer to how many? is certainly equally difficult for other sets of parameters $a, b$. However, some interesting information on how $a \cdot p_n + b$ behaves will be obtained for the first 10,000,000 primes $p_n$.

Let $m$ be the number of primes produced by $a \cdot p_n + b$ for $n \leq N$, i.e. if $a \cdot p_n + b$ is prime we can write $a \cdot p_n + b = q_m$ where $q_m$ is prime. The following Ubasic program lines have been used to determine whether $a \cdot p_n + b$ is prime or not:

```
while N<10000000
  p=nxtprm(p)
  inc N
  c=a*p+b
  if nxtprm(c-1)=c then inc m
wend
```

The program has been implemented for a set of values of the parameters $a$ and $b$. The result is shown in table 1. It is interesting to visualize the result. Because of the logarithmic behaviour of the distribution of primes it is reasonable represent $m/N$ as a

¹ Unsolved problem number 17.
function of \( \log_{10} N \) rather than as a function of \( N \). For this reason the value of \( m \) has been recorded during the computation for \( N=10, 10^2, 10^3, 10^4, 10^5, 10^6, \) and \( 10^7 \).

<table>
<thead>
<tr>
<th>( \alpha, \beta / N )</th>
<th>( 10 )</th>
<th>( 10^2 )</th>
<th>( 10^3 )</th>
<th>( 10^4 )</th>
<th>( 10^5 )</th>
<th>( 10^6 )</th>
<th>( 10^7 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.2</td>
<td>5</td>
<td>25</td>
<td>174</td>
<td>1270</td>
<td>10250</td>
<td>86027</td>
<td>738597</td>
</tr>
<tr>
<td>2.1</td>
<td>5</td>
<td>25</td>
<td>166</td>
<td>1221</td>
<td>9667</td>
<td>82236</td>
<td>711153</td>
</tr>
<tr>
<td>3.2</td>
<td>8</td>
<td>47</td>
<td>290</td>
<td>2380</td>
<td>18915</td>
<td>160127</td>
<td>1392733</td>
</tr>
<tr>
<td>4.1</td>
<td>3</td>
<td>21</td>
<td>145</td>
<td>1108</td>
<td>9314</td>
<td>78676</td>
<td>685069</td>
</tr>
<tr>
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<td>5</td>
<td>26</td>
<td>188</td>
<td>1492</td>
<td>12020</td>
<td>103010</td>
<td>903165</td>
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<td>2175</td>
<td>18019</td>
<td>153925</td>
<td>1342255</td>
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<tr>
<td>7.2</td>
<td>4</td>
<td>23</td>
<td>167</td>
<td>1288</td>
<td>10634</td>
<td>91232</td>
<td></td>
</tr>
</tbody>
</table>

The graphs in figure 1 show \( m/N \) (y-axis) as a function of \( \log_{10} N \) (x-axis) where \( m \) is the number of primes of the form \( \alpha p + \beta \) for \( n < N \). Figure 1b is an enlargement of figure 1a for large values of \( N \), \( (N \leq 10^7) \). The eight curves correspond to the following sets of parameters listed in the order in which the curves appear right hand side of the two figures: \((\alpha, \beta) = (3, 2), (6, 1), (5, 2), (8, 1)\).

<table>
<thead>
<tr>
<th>( B/N )</th>
<th>( 10 )</th>
<th>( 100 )</th>
<th>( 1000 )</th>
<th>( 10000 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
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<td>25</td>
<td>174</td>
<td>1270</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>27</td>
<td>170</td>
<td>126</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
<td>48</td>
<td>344</td>
<td>253</td>
</tr>
<tr>
<td>8</td>
<td>5</td>
<td>24</td>
<td>178</td>
<td>130</td>
</tr>
<tr>
<td>10</td>
<td>5</td>
<td>34</td>
<td>231</td>
<td>166</td>
</tr>
<tr>
<td>12</td>
<td>7</td>
<td>48</td>
<td>340</td>
<td>251</td>
</tr>
</tbody>
</table>
For \( n < N \)

<table>
<thead>
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<th>( 10^6 )</th>
<th>( 10^7 )</th>
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</thead>
<tbody>
<tr>
<td>86027</td>
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</tr>
<tr>
<td>82236</td>
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<td>1392733</td>
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<td>78674</td>
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<tr>
<td>103010</td>
<td>903165</td>
</tr>
<tr>
<td>153925</td>
<td>1342255</td>
</tr>
<tr>
<td>91232</td>
<td></td>
</tr>
</tbody>
</table>

(x-axis) where \( m \) is

enlargement of figure

the following sets of

parameters listed in the order in which the curves appear from top to bottom in the right-hand side of the two figures: \((a, b) = (3, 2), (6, 1), (5, 2), (7, 2), (1, 2), (2, 1), (4, 1)\) and

\((8, 1)\)

Table 2. Number of primes \( m \) generated by \( p = b \) for \( n < N \)

<table>
<thead>
<tr>
<th>( \frac{b}{n} )</th>
<th>( 10 )</th>
<th>( 100 )</th>
<th>( 1000 )</th>
<th>( 10000 )</th>
<th>( 100000 )</th>
<th>( 1000000 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>5</td>
<td>25</td>
<td>174</td>
<td>1270</td>
<td>10250</td>
<td>86227</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>27</td>
<td>170</td>
<td>1264</td>
<td>10214</td>
<td>85834</td>
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<td>6</td>
<td>7</td>
<td>48</td>
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<td>1303</td>
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<td>85866</td>
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<tr>
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<td>5</td>
<td>34</td>
<td>231</td>
<td>1682</td>
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</tr>
<tr>
<td>12</td>
<td>7</td>
<td>48</td>
<td>340</td>
<td>2515</td>
<td>20462</td>
<td>171618</td>
</tr>
</tbody>
</table>
The number of primes \( m \) in the sequence \( ap_n + b \) for \( n = 1 \) is shown in figure 2 for \( a = 1 \) and \( b = 2, 4, 6, 8, 10 \) and 12 when plotted against \( \log N \). The corresponding numerical results are given in table 2.

The general appearance of the graphs for various values of \( a \) and \( b \) is very similar. In particular figure 1b shows an arm running parallel to one another and in particular to the one for \( a = 1 \) which corresponds to \( p + 2 \). The width of the plot 2 for \( p + 2 \) which corresponds to prime twins for which there is a conjecture that there are infinitely many. This makes a reasonable conjecture: The progression \( ap_n + b \), \( (a,b)=1 \) contains infinitely many primes.

2. Prime Number Gaps

Smarandache asked how many primes there is in the progression \( ap_n + b \). The question is equivalent to 'how many twin primes are there?' A very stable conjecture that there are infinitely many we now have about their distribution and also about the distribution of primes \( g = p + 1 \). With a small change in the UBASIC program used, one can study the distribution of primes over gaps \( g = 2, 4, 6, \ldots \).

```plaintext
let p=3
while p<N
    q=p^2
    p=nextprime(p)
    u=(p-q)/2
    for i=1 to u
        if p(u)>0 then p(u)=q
        inc i(u)
    print "gap occurs in p(u)"
wend
```

This program was run for primes \( p < N = 2 \times 10^9 \). The result is the number of gaps \( g \) and \( p \) the prime number for which the gap occurs. All gaps \( g = 2 \) except 264, 278 and 286 are represented in figure 3. The gaps \( g = 4 \) and \( g = 8 \) occur much more frequently. This is illustrated in figure 3 which also shows that in fact the linear behaviour for all three types. The 'wild' behaviour should certainly correct itself if the range of primes in the study is increased.
The number of primes \( m \) in the sequence \( a_p+b \) for \( n = 1, 2, \ldots, N \) is illustrated in figure 2 for \( a = 1 \) and \( b = 2, 4, 6, 8, 10 \) and 12 where the ratio \( M=m/N \) is plotted against \( \log N \). The corresponding numerical results are given in table 2.

The general appearance of the graphs for various values of the parameters \((a,b)\) in \( a_p+b \) is very similar. In particular figure 1b shows an interesting picture of curves running parallel to one another and in particular to the one for \((a,b)=(1,2)\), that is the curve for \( p^2-2 \) which corresponds to prime twins for which we have the classical conjecture that there are infinitely many. This makes the following conclusion reasonable.

**Conjecture:** The progression \( a_p+b \), \((a,b)=1\) contains infinitely many prime numbers.

2. **Prime Number Gaps**

Smarandache asked how many primes there is in the progression \( a_p+b \). For \( a=1 \) and \( b=2 \) the question is equivalent to ‘how many twin primes are there?’ Since we have a very stable conjecture that there are infinitely many we now want to know something about their distribution and also about the distribution of other prime number gaps \( g=p_{n+1}-p_n \). With a small change in the Ubasic program used in the previous section we can study the distribution of primes over gaps \( g=2, 4, 6, \ldots \)

```bash
p=3
while p<=N
q=p
p=nxtprm(p)
if u=(p-q)/2/2 then u=q
if p(u)=0 then p(u)=q
wend

count the number of gaps p-q, store the smallest prime for which the gap occurs in p.
```

This program was run for primes \( p<=N=2.10^7 \). The result is shown in table 3, where \( f \) is the number of gaps \( g \) and \( p \) the prime number for which the gap first occurs, \( N=2.10^7 \). All gaps \( g<=292 \) except 264, 278 and 286 are represented in the table which is arranged so that gaps \( g=2 \) (mod 6), \( g=4 \) (mod 6) and \( g=0 \) (mod 6) are found in separate columns. Gaps \( g=0 \) (mod 6) occur much more frequently than the other two.

This is illustrated in figure 3 which also shows that in \( f \) as a function of \( g \) has a near linear behaviour for all three types. The “wild” behaviour for gaps>250 would certainly correct itself if the range of primes in the study were extended. The area
below the curves for \( g=2 \) (mod 6) and \( g=4 \) (mod 6) are equal as will be shown shortly. The curve for \( g=2 \) (mod 6) behaves very well while the one for \( g=4 \) (mod 6) shows an interesting ripple effect. In particular it shows a "bump" for \( g=70 \) which showed up already in the smallest sample \( N<10^5 \) for which \( g=70 \) first appeared. What causes this high frequency for \( g=70 \)?

For a prime number \( p \equiv 5 \) (mod 6) we have \( p \equiv 1 \) (mod 6). Let \( q \) and \( p \) be two consecutive primes forming the gap \( g=p-q \). We distinguish between the following cases:

- \( q \equiv 1 \) (mod 6) and \( p \equiv 1 \) (mod 6) \( \Rightarrow g \equiv 0 \) (mod 6)
- \( q \equiv 1 \) (mod 6) and \( p \equiv -1 \) (mod 6) \( \Rightarrow g \equiv 4 \) (mod 6)
- \( q \equiv -1 \) (mod 6) and \( p \equiv 1 \) (mod 6) \( \Rightarrow g \equiv 2 \) (mod 6)
- \( q \equiv -1 \) (mod 6) and \( p \equiv -1 \) (mod 6) \( \Rightarrow g \equiv 0 \) (mod 6)

A sequence of consecutive primes (with the first prime \( \equiv 5 \)) can be characterized by the shifts:

\[ ++, +\ldots+++, -+++-, ++-+++, -+-++- \]

The longest sequence of consecutive primes \( \equiv 1 \) (mod 6) for \( p<10^5 \) is of length 18:

\[
450988159, 450988177, 450988207, 450988231, 450988241, 450988261, 450988297, 450988333, 450988339, 450988381, 450988387, 450988399, 450988411, 450988423, 450988441, 450988471, 450988477, 450988567
\]

and the longest sequence of consecutive primes \( \equiv -1 \) (mod 6) for \( p<10^5 \) is of length 22:

\[
766319189, 766319201, 766319231, 766319249, 766319261, 766319273, 766319291, 766319339, 766319357, 766319363, 766319369, 766319423, 766319441, 766319453, 766319483, 766319507, 766319549, 766319573, 766319579, 766319621, 766319627
\]

Let the number of gaps \( \equiv 2 \) or \( \equiv 0 \) (mod 6) be \( f_2 \), \( f_4 \), and \( f_0 \) respectively for \( p<\text{N} \). Then we will have \( f_i=f_{i+1} \) if the last shift is ++ otherwise we will have \( f_i=f_{i+1} \).

In a simple model it would be reasonable to assume that, at an arbitrary point in the sequence of shifts, the probabilities of finding the next shift to be ++, +++, ++ or -- are equal, i.e. if we define \( F_2=f_2/(f_2+f_4+f_0) \), \( F_4=f_4/(f_2+f_4+f_0) \) and \( F_0=f_0/(f_2+f_4+f_0) \) we would have \( F_1=F_0=0.25 \) and \( F_2=0.5 \). This is not the case.

Before looking into this let's first consider a related question (notation \( f_i \) occur with the same frequency as primes \( \equiv 1 \) (mod 6)).

Table 4 shows a study of the number of primes \( f_i \) and \( f_{i-1} \) (mod 6) for primes less than \( 10^6 \) for \( k=1, 2, 3, \ldots, 9 \).

Within the range of this study we have \( f_i>f_{i-1} \). However, the rate is decreasing. Will eventually \( f_i>f_{i-1} \)?

We have proved that \( f_2=f_4 \) (assuming the last shift to be ++) relative frequencies defined through

\[ F_2=f_2/(f_2+f_4+f_0) \quad F_4=f_4/(f_2+f_4+f_0) \quad F_0=f_0/(f_2+f_4+f_0) \]

Again we have \( F_2=F_4 \) and of course \( F_0=1-2F_2 \). To study increase the number of consecutive primes \( p<10^6 \) the execution statistics was stopped for \( k=1, 2, \ldots, 9 \) to produce the data shown in Table 4.

Table 4. Number of primes \( \equiv -1 \) respectively \( 1 \) (mod 6)

<table>
<thead>
<tr>
<th>( k )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f_{-1} )</td>
<td>1</td>
<td>12</td>
<td>85</td>
<td>616</td>
<td>4805</td>
<td>39264</td>
<td>323263</td>
</tr>
<tr>
<td>( f_{-2} )</td>
<td>1</td>
<td>11</td>
<td>81</td>
<td>611</td>
<td>4785</td>
<td>39232</td>
<td>321914</td>
</tr>
<tr>
<td>( f_{-3} )</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td>5</td>
<td>20</td>
<td>32</td>
<td>189</td>
</tr>
<tr>
<td>( f_{-4} )</td>
<td>2</td>
<td>23</td>
<td>166</td>
<td>1227</td>
<td>9590</td>
<td>78496</td>
<td>664577</td>
</tr>
<tr>
<td>( r&lt;10^6 )</td>
<td>0</td>
<td>435</td>
<td>241</td>
<td>41</td>
<td>21</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 5. Prime number gap distribution (mod 6)!

<table>
<thead>
<tr>
<th>( k )</th>
<th>( \equiv 2 ) (mod 6)</th>
<th>( \equiv 4 ) (mod 6)</th>
<th>( \equiv 0 ) (mod 6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
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<td>0</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
<td>8</td>
<td>7</td>
</tr>
<tr>
<td>3</td>
<td>58</td>
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<tr>
<td>6</td>
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<tr>
<td>9</td>
<td>14107250</td>
<td>14107249</td>
<td>2263034</td>
</tr>
</tbody>
</table>
Before looking into this let's first consider a related question. Do primes (mod 6) or (mod 12) occur with the same frequency as primes = 1 (mod 6) or primes = 5 (mod 6) respectively? Table 4 shows a study of the number of primes in this range (f) occur within the range of this study we have f > 1. However, the ratio \( e^{-k} \) is decreasing. We will now study the relative frequencies defined through

\[ F = f \alpha(q+\beta) \]


Again we have \( F = f \alpha \) and of course \( F = 1 - 2 \alpha \). To study how \( F \) varies as we take \( \alpha \) larger, the number of consecutive primes \( p \) (the execution of the program for gap statistics was stopped for \( k = 1, 2, \ldots, 9 \) to produce the data shown in Table 5.

Table 5: Prime number gap distribution (mod 6) for primes < 10^10

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</table>

We have proved that \( F = f \alpha \). Assuming the last step to be \( + \), we will now study the prime number gap distribution (mod 6) for primes < 10^10.

Table 5: Prime number gap distribution (mod 6) for primes < 10^10

<table>
<thead>
<tr>
<th>k</th>
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<th>( \alpha ) (mod 12)</th>
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We have proved that \( F = f \alpha \). Assuming the last step to be \( + \), we will now study the prime number gap distribution (mod 6) for primes < 10^10.
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16
On Smarandache Functions

1. Smarandache - Fibonacci Triplets

We recall the definition of the Smarandache Function $S(n)$:

$S(n) = \text{the smallest positive integer such that } S(n) = S(n-1) + S(n-2)$

and the Fibonacci recurrence formula:

$F_n = F_{n-1} + F_{n-2} \ (n \geq 2)$

which for $F_0 = F_1 = 1$ defines the Fibonacci series.

We will concern ourselves with isolated occurrences of triplets $S(n)+S(n-1)+S(n-2)$ and pose the question: Are there infinitely many of them?

A straightforward search by applying the definition of the Smarandache function to consecutive integers was used to identify the first eleven such triplets, as given in table 1.

As often in empirical number theory this merely raises more questions and hence an ocean of integers. As can be seen from figure 1 the next triplet of $n$ such that $S(n)+S(n-1)+S(n-2)$ is impractical to be more precise.

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Figure 4. Relative frequency of prime gaps. $F_2 = F_3$ (thick line) and $F_0$ (thin line).

**Conclusion:** $F_2 < 0.45$ and $F_3 = F_4 > 0.27$ for primes $< 10^9$.

The approach to the values 0.5 and 0.25 that one would have expected is very slow and is slowed down with increasing $k$, - one realizes that the interval $8 < k < 9$ is ten times as large as the interval $0 < k < 8$. The data used is cumulative but even if we consider only the interval between $10^5$ and $10^6$ we have $F_7 = 12,659,767/45,680,669 = 0.2771$ compared to $F_6 = 0.2774$ for the whole interval between 0 and $10^9$.

**Question:** Given an arbitrarily small number $\delta > 0$, does a prime $p$ exist so that $F_p < 0.25 + \delta$ for all $p > p_0$?
Chapter II

On Smarandache Functions

1. Smarandache - Fibonacci Triplets

We recall the definition of the Smarandache Function $S(n)$:

$$S(n) = \text{the smallest positive integer such that } S(n)! \text{ is divisible by } n.$$ 

and the Fibonacci recurrence formula:

$$F_n = F_{n-1} + F_{n-2} \quad (n \geq 2)$$

which for $F_0 = F_1 = 1$ defines the Fibonacci series.

We will concern ourselves with isolated occurrences of triplets $n$, $n-1$, $n-2$ for which $S(n)! = S(n-1)! + S(n-2)!$ and pose the questions: Are there infinitely many such triplets? Is there a method of finding such triplets which would indicate that there are in fact infinitely many of them?

A straightforward search by applying the definition of the Smarandache Function to consecutive integers was used to identify the first eleven triplets [1] which are listed in Table 1. As often in empirical number theory this merely scratched the surface of the ocean of integers. As can be seen from figure 1 the next triplet may occur for a value of $n$ so large that a sequential search may be impractical and will not make us much wiser.

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Table 1. The first 11 Smarandache-Fibonacci triplets
However, an interesting observation can be made from the triplets already found. Apart from $n=26245$ the Smarandache-Fibonacci Triplets have in common that one member is two times a prime number while the other two members are prime numbers. This observation leads to a method to search for Smarandache Fibonacci triplets in which the following two theorems play a role:

I. If $n = ab$ with $(a,b) = 1$ and $S(a) < S(b)$ then $S(n) = S(b)$.

II. If $n = p^a$ where $p$ is a prime and $a \leq p$ then $S(p^a) = p$.

\[
\begin{array}{|c|c|c|}
\hline
\# & N & S(N) \\
\hline
1 & 4 & 4 \ast 3 \\
2 & 11 & 11 \ast 5 \\
3 & 121 & 22 \ast 5 \\
4 & 4902 & 43 \ast 29 \\
5 & 32112 & 223 \ast 197 \\
6 & 64010 & 173 \ast 46 \\
7 & 368140 & 233 \ast 82 \\
8 & 415664 & 313 \ast 167 \\
9 & 2091206 & 269 \ast 202 \\
10 & 2519648 & 1109 \ast 202 \\
11 & 4573053 & 569 \ast 106 \\
12 & 7783364 & 2591 \ast 202 \\
13 & 79269727 & 2861 \ast 2719 \\
14 & 136193976 & 3433 \ast 554 \\
15 & 321022289 & 7589 \ast 178 \\
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17 & 559199345 & 1129 \ast 662 \\
18 & 670994143 & 6491 \ast 838 \\
19 & 836250239 & 9859 \ast 482 \\
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28 & 3969415464 & 5801 \ast 1198 \\
29 & 3970638169 & 2066 \ast 643 \\
30 & 4652535626 & 3506 \ast 3007 \\
31 & 6079275799 & 3394 \ast 2837 \\
32 & 6493607750 & 3049 \ast 1262 \\
33 & 6964545435 & 2161 \ast 1814 \\
34 & 11329931930 & 3023 \ast 2026 \\
35 & 11695098243 & 12821 \ast 1294 \\
36 & 11777879792 & 2174 \ast 1597 \\
\end{array}
\]

Figure 1. The values of $n$ for which the first 31 Smarandache-Fibonacci triplets occur.

The search for Smarandache-Fibonacci triplets will be restricted to integers which meet the following requirements:

\[
\begin{align*}
n &= xp^a \text{ with } a \leq p \text{ and } S(x) < ax, & (1) \\
n-1 &= yq^b \text{ with } b \leq q \text{ and } S(y) < bq, & (2) \\
n-2 &= zr^c \text{ with } c \leq r \text{ and } S(z) < cr. & (3)
\end{align*}
\]
Table 2a. Smarandache-Fibonacci triplets

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Table 2b. Smarandache-Fibonacci triplets

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p.q and r are primes. We then have S(n)=ap, S(n-1)=bq and S(n-2)=cr. By subtracting (2) from (1) and (3) from (2) we get

\[ ap = bq + cr \]
\[ xp^t - yq^t = 1 \]
\[ yq^t - zr^t = 1 \]

The greatest common divisor (p^t, q^t)=1 obviously divides x and y. This is the condition for (5) to have infinitely many solutions. Such solutions are found using Euclid's algorithm and can be expressed as

\[ x = x_0 + q^t \cdot t \]
\[ y = y_0 - p^t \cdot t \]

where t is an integer and (x_0, y_0) is the principal solution.

Solutions to (5') are substituted in (6') in order to obtain integers z

\[ z = (yq^t - 1)/r^t \]

Solutions were generated for (a,b,c)=(2,1,1), (a,b,c)=(1,2,1) and the parameter t restricted to the interval -115t11. The results may be seen in table 2.

Since the correctness of these calculations is easily verified by verifying S(n), S(n-1) and S(n-2) these are given in table 3 for two larger extensions of table 2.

Table 3. Factorization of two Smarandache-Fibonacci triplets

| n | 16,736,688,950,356=2^3·3·19·1987·17,0072 | S9 |
| n-1 | 16,736,688,950,355=5·19·71·1399·1,741·6,277 | S9 |
| n-2 | 16,736,688,950,354=2·7·19·23·53·313·23,561 | S9 |

Conjecture:

There are infinitely many triplets n, n-1, n-2 such that S(n)=5 prime.
p, q and r are primes. We then have \( S(n) = ap, S(n-1) = bq \) and \( S(n-2) = cr \). From this and by subtracting (2) from (1) and (3) from (2) we get

\[
\begin{align*}
ap &= bq + cr \\
px^t - yq^t &= 1 \\
yq^t - zr^t &= 1
\end{align*}
\]  

(4) \hspace{2cm} (5) \hspace{2cm} (6)

The greatest common divisor \((p^t, q^t) = 1\) obviously divides the right hand side of (5). This is the condition for (5) to have infinitely many solutions for each solution \((p, q)\) to (4). Such solutions are found using Euclid's algorithm and can be written in the form:

\[
x = x_0 + q^t t, \quad y = y_0 - p^t t
\]

(5')

where \( t \) is an integer and \((x_0, y_0)\) is the principal solution.

Solutions to (5') are substituted in (6') in order to obtain integer solutions for \( z \).

\[
z = (yq^t - 1)x^t
\]

(6')

Solutions were generated for \((a, b, c) = (2, 1, 1), (a, b, c) = (1, 2, 1)\) and \((a, b, c) = (1, 1, 2)\) with the parameter \( t \) restricted to the interval \(-11 \leq t \leq 11\). The result is shown in table 2. Since the correctness of these calculations are easily verified from factorizations of \( S(n), S(n-1) \) and \( S(n-2) \) these are given in table 3 for two large solutions taken from an extension of table 2.  

<table>
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<th>( S(n-1) )</th>
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Table 3. Factorization of two Smarandache-Fibonacci triplets

**Conjecture:**

There are infinitely many triplets \( n, n-1, n-2 \) such that \( S(n) = S(n-1) + S(n-2) \).
Questions:

1. It is interesting to note that there are only 7 cases in table 2 where \( S(n-2) \) is two times a prime number and that they all occur for relatively small values of \( n \). Which is the next case?

2. The solution for \( n=26245 \) stands out as a very interesting one. Is it a unique case or is it a member of a family of Smarandache-Fibonacci triplets different from those studied here?

References:


2. Radu's Problem

For a positive integer \( n \), the Smarandache function \( S(n) \) is defined as the smallest positive integer such that \( S(n)^z \) is divisible by \( n \). Radu [1] noticed that for nearly all values of \( n \) up to 4800 there is always at least one prime number between \( S(n) \) and \( S(n+1) \) including possibly \( S(n) \) and \( S(n+1) \). The exceptions are \( n=224 \) for which \( S(n)=8 \) and \( S(n+1)=10 \) and \( n=2057 \) for which \( S(n)=22 \) and \( S(n+1)=21 \). Radu conjectured that, except for a finite set of numbers, there exists at least one prime number between \( S(n) \) and \( S(n+1) \). The conjecture does not hold if there are infinitely many solutions to the following problem.

*Find consecutive integers \( n \) and \( n-1 \) for which two consecutive primes \( p_k \) and \( p_{k-1} \) exist so that \( p_k < \min(S(n),S(n-1)) \) and \( p_{k-1} > \max(S(n),S(n-1)) \).*

Consider

\[
  n+1 = x p_k^z
\]  \hspace{1cm} (1)

and

\[
  n = y p_{k-1}^z
\]  \hspace{1cm} (2)

where \( p_k \) and \( p_{k-1} \) are consecutive prime numbers. Subtract (2) from (1).

\[
  x p_k^z - y p_{k-1}^z = 1
\]  \hspace{1cm} (3)
The greatest common divisor \( (p^s, p_{-1}^t) = 1 \) divides the right hand side of (3) which is the condition for this diophantine equation to have infinitely many solutions. We are interested in positive integer solutions \((x, y)\) such that the following conditions are met.

\[
S(n+1) = sp, \ \text{i.e.} \ S(x) < sp
\]
\[
S(n) = sp_{-1}, \ \text{i.e.} \ S(y) < sp_{-1}
\]

In addition we require that the interval

\[
sp^t < q < sp_{-1}^t
\]

is prime free, i.e. that \( q \) is not a prime.

Euclid's algorithm is used to obtain principal solutions \((x_0, y_0)\) to (3). The general set of solutions to (3) is given by:

\[
x = x_0 + p_{-1}^t s, \quad y = y_0 - p^t t
\]

with \( t \) an integer.

These algorithms were implemented for different values of the parameters \( d=p_{-1}-p_s \), \( s \) and \( t \). The result was a very large number of solutions. Table 4 shows the 20 smallest (in respect of \( n \)) solutions found. There is no indication that the set would be finite.

One pair of primes may produce several solutions.

Within the limits set by the design of the program the largest prime difference for which a solution was found was \( d=42 \) and the largest exponent which produced solutions was \( s=4 \). Some numerically large examples illustrating these facts are given in table 5.

To see the relation between these large numbers and the corresponding values of the Smarandache function in table 5 the factorizations of these numbers are given below:

\[
118229366471229578483018 = 2 \cdot 23 \cdot 193 \cdot 431 \cdot 1612781193
118229366471229578483017 = 509 \cdot 3253 \cdot 16128239
11157906497858100263736883634 = 2 \cdot 7 \cdot 37 \cdot 5667 \cdot 55333
11157906497858100263736883635 = 3 \cdot 5 \cdot 11 \cdot 19 \cdot 16433 \cdot 55333
17549865213221116241350236227 = 3 \cdot 11 \cdot 3 \cdot 107 \cdot 12671 \cdot 55333
17549865213221116241350236226 = 2 \cdot 23 \cdot 37 \cdot 41 \cdot 743 \cdot 55333
27032997592120253634707051822848570391314 = 2 \cdot 3 \cdot 47 \cdot 1289 \cdot 2017 \cdot 119983 \cdot 167441
27032997592120253634707051822848570391313 = 37 \cdot 23117 \cdot 24517 \cdot 38303 \cdot 167443
\]
Table 4. The 20 smallest solutions which occurred for s=2 and d=2

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Table 5. Four numerically large solutions

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It is also interesting to see which are the nearest smaller Pk and nearest bigger Pk-1 primes to S1 = Min(S(n),S(n+1)) and S2 = Max(S(n),S(n+1)) respectively. This is shown in table 6 for the above examples.
3 and d=2

Table 6. $P_k < S_1 < S_2 < P_{k+1}$

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Conclusion:
There are infinitely many intervals $\{\text{Min}(S(n), S(n+1)), \text{Max}(S(n), S(n+1))\}$ which are prime free.

References:
I. M. Radu, Mathematical Spectrum, Sheffield University, UK, Vol. 27, No. 2, 1994/5, p. 43.

3. The Smarandache Cell Function

Definition: For a positive integer $n$ the Smarandache cell function of order $k$ is defined through $^1$

$$S_k(n) = m$$

where $m$ is the smallest positive integer for which $n$ divides $m^k$.

In the study of this function we will make frequent use of the ceil function defined as follows:

$$\lceil x \rceil$$

the smallest integer not less than $x$.

The following properties follow directly from the above definitions:

1. $S_1(n) = n$
2. $S_k(p^a) = p^{a^k}$ for any prime number $p$.
3. For distinct primes $p, q, \ldots r$ we have $S_k(p^aq^b\cdots r^k) = p^{a^k}q^{b^k}\cdots r^{k^k}$.

Theorem I. $S_k(n)$ is a multiplicative function.

---

$^1$ This function has a great resemblance to the Smarandache function. Its definition was proposed by K. Kashiwara (Japan) and conveyed to the author by R. Muller.
A function \( f(n) \) is said to be multiplicative if for \((n_1, n_2) = 1 \) it is true that \( f(n_1 n_2) = f(n_1) f(n_2) \). In our case it follows directly from (3) that if \((n_1, n_2) = 1 \) then \( S_k(n_1 n_2) = S_k(n_1) S_k(n_2) \).

However, consider \( n = n_1 n_2 \) when \((n_1, n_2) = 1 \). In a simple case let \( n_1 = m_1 \alpha^p \) and \( n_2 = m_2 \beta^q \) with \((m_1, m_2) = 1 \) we then have \( S_k(n) = S_k(m_1) S_k(m_2) \alpha^{p \gamma} \beta^{q \delta} \) which differs from \( S_k(n_1) S_k(n_2) \) whenever \( \lfloor (\alpha + \beta) k \rfloor = \lfloor \alpha k \rfloor + \lfloor \beta k \rfloor \). In fact one easily proves that \( \lfloor (\alpha + \beta) k \rfloor = \lfloor \alpha k \rfloor + \lfloor \beta k \rfloor \) or \( \lfloor (\alpha + \beta) k \rfloor = \lfloor \alpha k \rfloor + \lfloor \beta k \rfloor - 1 \).

**Theorem II.** \( S_k(n) \) divides \( S_k(n) \)

Express \( n \) in prime factor form \( n = p_1^{a_1} p_2^{a_2} \ldots p_r^{a_r} \) and apply (3). We then see that all prime powers in \( S_k(n) \) are less than or equal to those of \( S_k(n) \), i.e. \( S_k(n) \mid S_k(n) \).

**Theorem III.** For sufficiently large values of \( k \) we have \( S_k(n) = \prod p_i \) where the product is taken over all distinct primes \( p_i \) of \( n \).

By extending the argument in theorem II we have that, if \( j = \max(\alpha, \beta, \ldots, \delta) \) then \( S_k(n) = p_j \cdot r \) for \( k \geq j \).

**Corollary 1.** \( S_k(p) = p \) for any prime number \( p \).

**Corollary 2.** If \( n \) is square free then \( S_k(n) = n \).

**Theorem IV.** \( k \) exists so that \( S_k(n!) = p^\delta \), where \( p \) is the largest prime dividing \( n \).

\( p^\delta \) denotes the product of all primes less than or equal to \( p \). Let’s write \( n! \) in prime factor form \( n! = 2^{s_2} 3^{s_3} \ldots p^r \), where \( \alpha > \beta > \ldots > \gamma \). In order to apply theorem III we need to find \( \alpha \). Consider 1-2-3-4-5-6-\ldots \( n \). This product contains \( \lfloor n/2 \rfloor \) even integers, \( \lfloor n/4 \rfloor \) multiples of 4, etc and finally \( \lfloor n/2^j \rfloor \) multiples of \( 2^j \), where \( 2^j n! < 2^{j+1} \). \( \delta \) is determined by \( \delta = \lceil \log n / \log 2 \rceil \). From this we find that \( S_k(n!) = p^\delta \) for

\[
k = \alpha = \sum_{r=1}^{\delta} \left\lfloor \frac{n!}{2^r} \right\rfloor
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### Concepts and Applications

1. IL changes the way you
2. extend your existing Java
3. knowledge
4. Servlets, JSP, and Swing
5. access strategies that
6. Information using XSLT
7. XML DOM and how to
8. IL into Java database
9. SAX API in Java

Vare including:

- server
- files from the book
Calculations. Calculation of \( S_k(n) \) for \( n < 1000 \) were carried out in Ubasic, which has a built in ceil function. The result is shown in table 7. Since \( S_k(n) = n \) for square free numbers these have been excluded from the table. When \( S_k(n) \) is square free the entries for larger values of \( k \) become repetitive. Instead of repeating these values the corresponding spaces in the table have been left blank.

4. The Smarandache Pseudo Function \( Z(n) \)

Definition\(^2\): \( Z(n) \) is the smallest positive integer \( m \) such that \( 1+2+\ldots+m \) is divisible by \( n \).

Alternative formulation: For a given positive integer \( n \), \( Z(n) \) equals the smallest positive integer \( m \) such that \( m(m+1)/2n \) is an integer.

The following properties follow directly from the definition:

1. \( Z(1)=1 \)
2. \( Z(2)=3 \)
3. For any odd prime number \( p \), \( Z(p)=p-1 \)
4. By extension of (3) we have \( Z(p^k)=p^k-1 \)
5. In the special case \( n=2^k \) we have \( Z(2^k)=2^{k-1}-1 \)

Calculation of \( Z(n) \)

We need to find \( m \) so that \( m(m+1)=2nk \) has a positive integer solution for the smallest possible positive value of \( k \).

\[
m = \frac{-1 + \sqrt{1 + 8kn}}{2}
\]

For a given value of \( n \) the smallest square \( 1+8kn \) is found by executing the following program lines in Ubasic where effective use of the ISQRT(x) has been made:

\(^2\) Definition by K.Kashiwara (Japan) conveyed to the author by R. Muller, Erhus University Press, USA.
10 INPUT "n ",n
20 k=0
30 inc k
40 x=1+8*k*n
50 if x<=(isqrt(x))*2 then goto 30
60 goto 50
70 end

The complete program has been implemented for n ≤ 1000. The result is displayed in table 8.

**Theorem:** If n = pq, where p and q are two distinct primes with g = q-p, then

\[ Z(n) = \min(p(qk+1)/g \text{ where } pk+1=0 \pmod{g}, q(pk-1)/g \text{ where } pk-1=0 \pmod{g}) \]

**Proof:**
We have to consider three cases:
1. \( p \mid m \) and \( q \mid (m+1) \) which, since we assume \( q > p \), we distinguish from
2. \( p \mid (m+1) \) and \( q \mid m \)
3. \( pq \mid (m+1) \)

Case 1. Consider \( px = m \) and \( qy = m+1 \) which together with \( g = q-p \) gives

\[ p(x+y) = gy-1 \quad (1) \]

Since we must have \( p \mid (gy-1) \) we can put \( gy-1 = pk \) where \( k \mid (x-y) \). Our solution for \( y \) then becomes

\[ y = (pk+1)/g \text{ with } pk+1=0 \pmod{g} \quad (2) \]

Inserting this in (1) results in

\[ p(x-(pk+1)/g) = pk \]

from which

\[ x = (qk+1)/g \quad (3) \]

which we insert in \( m = px \) to obtain

\[ m = p(qk+1)/g \text{ where } k \text{ is determined through } pk+1=0 \pmod{g} \]

ICEPTS AND .ICATIONS
changes the way you tend your existing Java knowledge
instead of JSP, and Swing for strategies that formation using XSLT
ML DOM and how to into Java database

\[ \text{C} \backslash \text{API in Java} \]

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Table 8d. Z(n) for n ≤ 1000, n non-prime
Case 2. Consider $p_x = m+1$ and $q_y = m$ together with $g = q - p$. Repeating the calculation in case 1 results in:

$$m = q(p_k-1)/g$$ where $k$ is determined through $pk-1 = 0 \pmod g$

Case 3. Consider $m = pq-1$. We must have $pq-1 < p(qk+1)/g$ to beat case 1. This inequality can be written $p(q(k+1)/g - q > g$ where $g < k$ and $g > 2$ from which we see that this inequality is impossible. The argument is similar to reject case 3 in comparison with case 2. $m$ is therefore given by the minimum value for $m$ as calculated in cases 1 and 2, i.e.

$$Z(n) = \min(p(qk+1)/g \text{ where } pk+1 = 0 \pmod g, q(pk-1)/g \text{ where } pk-1 = 0 \pmod g)$$

**Corollary:** $Z(n)$ is not a multiplicative function.

The above theory has been used to calculate $Z(pq)$ for a few prime number gaps $g = q - p$. The result is shown in table 9.

### Table 9a. $Z(n)$ for the first 10 prime gaps $g = 10$ and $g = 30$

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### Table 9b. $Z(n)$ for the first 10 prime gaps $g = 20$

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Table 9b. $z(n)$ for the first 10 prime gaps $g=20$ and $g=40$

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Concepts and Plications

IL changes the way you extend your existing Java knowledge

- Servlets, JSP, and Swing access strategies that
- information using XSLT
- XML DOM and how to
- ML into Java databases
- SAX API in Java

Server examples from the book
Chapter III

Loops and Invariants

In many practical as well as theoretical processes we repeat the same operation on an object again and again in order to arrive at a final result, or sustain a certain state or maybe simply to see what is going to happen. Each time a hammer hits a nail the nails sinks a bit deeper until it can sink no further. This repeated operation has resulted in an invariant state. A different situation occurs in an engine when energy is used to make a piston perform cycles or loops. Many similar phenomena occur when repeating processes on numbers through iterations. Before going into the problems of this section let us consider the iteration process itself.

Let \( I(n) \) define an operation to be carried out on \( n \). If we apply this operation to \( I(n) \) itself we say that we perform an iteration and could write this as \( I(I(n)) \). After a number of iterations we could have something like \( I(I(\ldots(I(n))\ldots)) \), which we will write \( I_q(n) \) to indicate the \( k \)th iteration. Alternatively we can use \( n_0 \) to denote the result of the \( k \)th iteration and \( n_0 \) to denote the starting value. We then have

\[
\begin{align*}
 n_1 &= I(n_0), \\
 n_2 &= I(n_1), \\
 & \quad \ldots \quad \\
 n_{k-1} &= I(n_k)
\end{align*}
\]

Let us apply this to a simple case where \( I(n) \) is defined through \( I(n) = 1/n \). If we take \( n_0 = 1 \) then \( n_0 = 1 \) for all \( k \), i.e. the result of the iteration is invariant. If we take \( n_0 = 2 \) then \( n_0 = 1/2 \) for odd values of \( k \) and \( n_0 = 2 \) for even values of \( k \), i.e. we have an iteration resulting in a loop of length 2. If we apply the iteration process to \( I(n) = n^2 \) then the result will be forever increasing and we say that the iteration is divergent.

Since we are dealing with a very important concept which has attracted a lot of attention some of the topics in this section have been dealt with before. In particular Perfect Digital Invariants [1], which has recently been reactivated under another name “Steinhaus’ problem” [2]. However, all results presented here have been generated in recent studies carried out by the author and have been retained even though some of them may duplicate earlier results. This proved necessary in order to arrive at a consistent presentation and some new results. J. S. Madachy has made me aware of more literature on this interesting topic some of which is listed in the references.

1. Perfect Digital Invariants and Related Loops

For an arbitrary positive integer

\[
N_k = a_n10^n + a_{n-1}10^{n-1} + \ldots + a_110 + a_0
\]

we define \( N_{k+1} = I(N_k) \) through

\[
I(N_k) = a_0^q + a_1^q + a_2^q + \ldots + a_n^q + a_0^q, \text{ where } q \in \mathbb{N}, q > 0
\]

For a given positive integer \( n_0 \) the iteration process \( n_1 \) continued until one of the following situations is reached:

\begin{align*}
&\text{I. } I(n_k) = n_k & \text{Perfect Digital Invariant} \\
&\text{II. } I(n_{k-1}) = n_k & \text{Loop of length L} \\
&\text{III. } I(n_k) \to \infty & \text{Divergent}
\end{align*}

Case III is impossible. It follows from the following determinate limit for each value of \( q \). Let us consider the largest possible value of \( a \), namely 9.

For all \( N > N_{\text{max}} \) let \( u \) be the largest value of \( n \) for which

\[
10^n - 1 - u \cdot 9^9 < 0 \quad \text{while} \quad 10^{n+1} - 1 - (u+1) \cdot 9^9 > 0
\]

then there is a smallest positive integer \( a \), such that

\[
a \cdot 10^9 + 10^9 - 1 - a \cdot 9^9 > 0
\]

This gives \( N_{\text{max}} = (a+1) \cdot 10^9 - 1 \) as an upper limit for solutions by looking for a smaller value than 9 for the second most would give more complications than benefits in computer work. This proves that the iteration process does not diverge for iterations larger than \( N_{\text{max}} \), a previously assumed value must loop or collapsing on an invariant.

Only a small subset of all integers \( < N_{\text{max}} \) needs to be used in the search program. The following two input criteria greatly reduce the execution time.
1. Perfect Digital Invariants and Related Loops

For an arbitrary positive integer
\[ N_k = a_0 10^0 + a_1 10^1 + \ldots + a_{10} 10^{10} \]
we define \( N_{k-1} = I(N_k) \) through
\[ I(N_k) = a_0^3 + a_1^3 + \ldots + a_{10}^3 \]
where \( q \in \mathbb{N}, q \geq 2 \).

For a given positive integer \( n_0 \) the iteration process \( n_1 = I(n_0), n_2 = I(n_1), \ldots \) is continued until one of the following situations is reached:

- **I.** \( I(n_k) = n_k \) \hspace{2cm} Perfect Digital Invariant (PDI)
- **II.** \( I(n_{k-1}) = n_k \) \hspace{2cm} Loop of length \( L \)
- **III.** \( I(n_k) \to \infty \) \hspace{2cm} Divergent

Case III is impossible. It follows from the following determination of the upper search limit for each value of \( q \). Let us consider the largest possible \( n \)-digit number \( N = 999...99 = 10^n-1 \). We have \( I(N) = n \cdot 9^q \) and need to determine \( N_{\text{max}} \) so that \( N - I(N) > 0 \) for all \( N > N_{\text{max}} \). Let \( u \) be the largest value of \( n \) for which \( 10^u - 1 - n \cdot 9^q < 0 \), i.e.
\[ 10^u - 1 - n \cdot 9^q < 0 \quad \text{while} \quad 10^{u-1} - 1 - (u+1) \cdot 9^q > 0 \]
then there is a smallest positive integer \( a, 1 \leq a \leq 9 \), such that
\[ a \cdot 10^u + 10^u - 1 - a^q - u \cdot 9^q > 0 \]

This gives \( N_{\text{max}} = (a+1) \cdot 10^{u-1} - 1 \) as an upper limit for solutions. \( N_{\text{max}} \) could be improved by looking for a smaller value than 9 for the second most significant figure but this would give more complications than benefits in computer implementation. That \( N_{\text{max}} \) exists proves that the iteration process does not diverge since after a number of iterations larger than \( N_{\text{max}} \) a previously assumed value must be repeated completing a loop or collapsing on an invariant.

Only a small subset of all integers \( < N_{\text{max}} \) needs to be used as input numbers in a search program. The following two input criteria greatly reduce the computer execution time.

- Server
- AX API in Java

See the book for more details.
1. The order in which digits occur in an input number is of no importance. \( N_0 = 2337 \) will give the same result as \( N_0 = 3732 \). A number whose digits are a permutation of an already used input number will therefore be rejected. As an example take \( q=5 \) for which \( N_{max} = 299999 \). In this case an input number needs to have maximum 6 digits of which there can be at most 6 ones or twos and a maximum of 5 of the other digits.

2. Input criterion number 1 is used so that the search always proceeds from a smaller to a larger input number. If for any input number \( N_0 \) we have \( I(N_0) < N_0 \) then the iteration process is aborted since \( I(N_0) \) has either been dealt with before or doesn’t meet criterion number 1.

Complete solutions were calculated for \( q = 2, 3, 4, \ldots, 15 \). Apart from the trivial case \( I(N)=1 \) these solutions are given in table 1 together with the upper search limit for each \( q \). The longest loop is of length 381 for \( q = 14 \). There are no Perfect Digital Invariants (PDIs), i.e. solutions to \( N = I(N) \) for \( q=2, q=12 \) and \( q=15 \).

References:

3. Dean Morrow, *Journal of Recreational Mathematics*, pgs 9-12, Vol. 27.1

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Table 1a. PDIs and related loops

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Table 1c. PDIs and related loops

Michael Decker is pleased to provide this corrected table.
### Table 1d. PDIs and related loops

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### 2. The Squambling Function

The squambling function U(N) originates from *The Penguin Book of Curious and Interesting Numbers*, page 169. In its present form the result was given by G. Samson in the November 1995 issue of *Personal Computing*.

For a given integer \( N = a_0 + a_1 10^1 + \ldots + a_{n} 10^n > 10^5 \), \( a_0 \neq 0 \), \( a_k \geq 0 \), given by

\[
U(N) = a_n n^2 + a_{n-1} n^{1} + \ldots + a_0^2
\]

Iterating the squambling function will result in an inverse squambling problem where there are no starting values for which to consider

\[
U(N) = a_n n^2 + a_{n-1} n^{1} + \ldots + a_0^2 < 9^2 + 9^1 + \ldots + 9^0
\]

If \( N_0 \) exists for which \( U(N) < N_0 \) for all \( N > N_0 \) then the process converges (or goes into a loop). \( N_0 \) exists and is found by \( 10^4 \) from the inequality below

\[
U(N) < 81 (9^{n-1}-1)/8 < 10^5 < N
\]

If \( N_0 \) exists for which \( U(N) < N_0 \) for all \( N > N_0 \) then the process converges (or goes into a loop). \( N_0 \) exists and is found by \( 10^4 \) from the inequality below

\[
U(N) < 81 (9^{n-1}-1)/8 < 10^5 < N
\]

More detailed considerations may considerably lower this upper limit. The result of a search for squam-invariant integers up to \( 10^6 \) is summarized in table 2, where \( L \) is the length of the loop, which equals 1 for an integer in the loop, \( N \) the largest number in the loop, and \( M \) found for initiating integers less than \( 10^5 \).

<table>
<thead>
<tr>
<th>M</th>
<th>1</th>
<th>43</th>
<th>63</th>
<th>47016</th>
<th>542186</th>
<th>126</th>
</tr>
</thead>
<tbody>
<tr>
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<td>542186</td>
<td>478</td>
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<tr>
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<td>613722</td>
<td>4617</td>
<td>125</td>
<td>6</td>
<td></td>
</tr>
</tbody>
</table>

---

For table 2: Squam-loops and squam-invariant integers.
2. The Squambling Function

The squambling function \( U(N) \) originates from *The Penguin Dictionary of Curious and Interesting Numbers*, page 169. In its present form the problem was formulated by G. Samson in the November 1995 issue of *Personal Computer World*:

For a given integer \( N = a_0 + a_1.10 + \ldots + a_{10}10^a > 10^a, a_r = 0 \), the squambling function is given by

\[
U(N) = a_0^2 + a_1^2 + \ldots + a_{10}^2.
\]

Iterating the squambling function will result in an invariant or a loop. As in the previous problem there are no starting values for which the process diverges. To see this consider

\[
U(N) = a_0^2 + a_1^2 + \ldots + a_{10}^2 < 9^2 + 9^2 + \ldots + 9^2 = 81(9^{10-1})/8
\]

If \( N_0 \) exists for which \( U(N) < N_0 \) for all \( N > N_0 \) then the squambling function must return to one of its previous values after a sufficiently large number of iterations, i.e. the process converges (or goes into a loop). \( N_0 \) exists and is estimated to be less than \( 10^{19} \) from the inequality below.

\[
U(N) < 81(9^{10-1})/8 < 10^9 < N
\]

More detailed considerations may considerably lower this upper limit for \( N_0 \), but in any case the upper limit is so large that a complete search for invariants and loops is hardly feasible. The result of a search for squam-invariants and squam-loops for integers up to \( 10^7 \) is summarized in table 2, where \( L \)=the total number of numbers in the loop (=length of the loop, which equals 1 for an invariant), \( M \)=the smallest number in the loop, \( N \)=the largest number in the loop and \( Q \)=the number of loops found for initiating integers less than \( 10^9 \).

<table>
<thead>
<tr>
<th>( L )</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>1</th>
<th>3</th>
<th>8</th>
<th>105</th>
</tr>
</thead>
<tbody>
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<td>63</td>
<td>47016</td>
<td>542186</td>
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<td>579</td>
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<td>125</td>
<td>6</td>
<td>13</td>
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<td>214018</td>
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</tbody>
</table>

Table 2. Squam-loops and squam-invariants
If there are more invariants and loops then they must have a smallest element greater than $10^n$, or more precisely a smallest element which is larger than the upper limit for a previous search. This has been used in an extended search for starting integers up to 50,000,000. No new loops were found. Question: Are there any?

Let us finally consider what happens if we reduce the powers to which each digit is raised by one. This makes the process a fortiori convergent as can be seen from the arguments used previously. The iteration process was examined for starting integers up to $10^7$. No loops were found. Apart from iterations ending on one-digit invariants a number of other invariants were found.

Table 3. The number of start integers $Q$ below $10^9$ resulting in an invariant $I$

<table>
<thead>
<tr>
<th>I</th>
<th>89</th>
<th>135</th>
<th>175</th>
<th>518</th>
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<th>1306</th>
<th>1676</th>
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<tbody>
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<td>319</td>
<td>128</td>
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<td>20</td>
<td>27</td>
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</table>

Question: Are there no loops in this case?

3. Wondrous Numbers

This study deals with the extended Wondrous Numbers Conjecture stated as follows by B.C. Wiggin [1]:

*Any integer $n \geq (D-1)$ may be directed through a series of iterations as a function $n = R \pmod{D}$. Starting with $n_0$ the series of iterated Wondrous Numbers are $n_1, n_2, \ldots, n_k, \ldots$*

**Definition:** Let $n_k \equiv R_k \pmod{D}$

- If $R_k = 0$ then $n_{k+1} = n_k / D$
- If $1 \leq R_k \leq D - 2$ then $n_{k+1} = n_k D + 1 - R_k$
- If $R_k = D - 1$ then $n_{k+1} = n_k (D + 1) + 1$

*It is conjectured that the series ultimately converges to $n < D$.*

This study will show that the above conjecture does not hold.

To examine the behaviour of the above process assume that for a given $n_k$ we have

$n_k \equiv 0 \pmod{D}$

in which case

$n_{k+1} = n_k / D; \quad n_k = D n_{k-1}$

Let

$n_{k-1} = R_{k-1} \pmod{D}$

We have to consider three cases

**Case I**

If $R_{k-1} = 0$

then

$n_{k-2} = n_{k-1} / D; \quad n_{k-2} = n_k / D^2$

**Case II**

If $0 < R_{k-1} < D - 2$

then

$n_{k-2} = n_{k-1} (D + 1) - R_{k-1}$

From this we see that $n_{k-2} \equiv 0 \pmod{D}$ and substituting for

$n_{k-2} = n_k (1 + 1 / D)$

If $R_{k-1}$ stays within the above limits for $l=1, 2, \ldots, m$ then

$n_{k-2m} < n_k (1 + 1 / D)^m$

**Case III**

If $R_{k-1} = D - 1$

As in case II $n_{k-2} \equiv 0 \pmod{D}$ but
Case I

If \( R_0 > 0 \), then we have \( m = \frac{R_0}{D_0} \).

Case II

If \( R_0 = 0 \), then we have \( m = \frac{D_0}{R_1} \).

Case III

If \( R_0 < 0 \), then we have \( m = \frac{D_0}{R_1} \).

We have to consider these cases.

Let

\[ n_1 = R_0 \]

in which case

\[ n_2 = n_0 D_0 \]

in which each digit is

can be seen from the

one digit maintains a

upper limit for

multiple integers up to

27

is not a multiple of a

integers.

\[ D_0 \]

is a whole number.

\[ m \]

is a whole number.

\[ R_0 \]

is a whole number.

\[ D_0 \]

is a whole number.

\[ R_1 \]

is a whole number.

\[ m \]

is a whole number.
\[ n_{k-1} < n_k (1 + 1/D) + 1 \]  \hspace{1cm} (4) 

For the series to diverge the case \( R=0 \) must occur with a frequency which is much higher than the expected \( 1/D \) (on the average). In a worst case scenario let us assume that \( R \neq 0 \) so many times that it balances the effect of the occurrence of case \( L \). Ignoring (4) which can be considered as compensated by ignoring \( R_{k-1} \) in (3) we then have:

\[ n_k (1 + 1/D)^2 / D^2 = \lambda \]

For \( D = 11 \) this gives \( m \approx 56 \) just to hold the balance. If remainders \( R \) are evenly distributed then \( m \) would vary around 10. It is very safe to say that the series does not diverge. However, it does not always converge to \( n < D \) as conjectured. Brendan Woods [2] reported that she failed to get termination with \((D,N) = (13,70), (14,75), (58,59), (82,83) \text{ and } (198,199)\). The reason is that these cases produce loops as shown in table 4. \( N_0 = 199 \), \( D = 198 \) produces a very long loop of length 2279 requiring 2499 iterations.

Charles Ashbacher [3] investigated all divisors in the interval \( 3 \leq D \leq 12 \) for all initial values \( n_0 < 14000000 \). Even with the most effective programming and up to date equipment this is a very impressive piece of work. In all cases iterations terminated on a one-digit number. He listed those cases for \( D=11 \) and \( D=12 \) where more than 1000 iterations were required before the terminal value was reached and made the following observation "...for certain series of ascending \( n \) the number of cycles descended in steps of 2 contrary to the expected behaviour that the number of cycles rises with the size of the number". He adds that no justification has been found for this and challenges readers to further explore this behaviour. The remainder of this section will be devoted to an explanation of this mystery though an analysis of the case \( D = 11 \). A similar analysis to the one below has been carried out for \( D = 12 \) with similar results.

Since all iterations result in a one-digit terminal value all cases with \( n_0 < 14000000 \) which require more than 1000 iterations can be classified according to their terminal value. This is done in the column marked \( p \) in table 6, which contains Ashbacher's table as a subset. There are only three different terminal values 3, 7 and 8.

Table 5 shows the 180 first iterations for \( n_0 = 1345680 \), which is the first entry in Ashbacher's table. Table 5 confirms the above theory that at least every second \( R \) is zero. This explains the step 2 decrease with ascending \( n \) since this causes the terms to oscillate. In general every second term was out of bounds of the investigation which was limited to \( n_0 < 14000000 \). These cases with long cycles are due to long "freak"
oscillate. In general every second term was out of bounds of the investigation which was limited to $n_0 < 14000000$. These cases with long cycles are due to long "freak" sequences if iterations with remainders $R=0$. Let's apply (3') to the first occurrence of $R=0$ for even $k$ in table 5. $n_1 = 16148154$, $n_{10} = 167493499$.

$$n_{10} < 16148154/(1 + 1/11)^{32}/121 \approx 167494096$$

This, as is seen, is a very good approximation.

| Table 4. Loops instead of anticipated convergence |
|----------------|----------------|----------------|----------------|----------------|
| $n$        | $n_0$        | Loop starts | First term of | Loop finishes |
| 13         | 70           | 92           | 1911          | 162           | 71           |
| 14         | 75           | 13           | 166           | 91            | 79           |
| 58         | 59           | 19           | 4002          | 555           | 537          |
| 82         | 83           | 158          | 13120         | 925           | 773          |
| 198        | 199          | 221          | 61300         | 2499          | 2279         |

We also see from table 5 that it contains a sequence of numbers which also occur in table 6, i.e. the initiating value $n_0 = 1345680$ is the "grandparent" of a whole family of larger initiating values having the same terminal value 8. However, table 6 also contains initiating values with terminal value 8 which do not occur in table 5. At some stage, however, these will merge with the iteration process for the "grandparent". These cases have been identified and labeled in column $c$ ("child") in table 6. These "children" have been used as starting numbers for iteration processes to see at which point they will join the "grandparent" iteration. The result is shown in tables 7 and 8, where $K_c$ is the number of iterations for the child (which may have a number of children of its own as can be seen from table 6), and $K_p$ is the number of iterations for the grandparent before merging occurs. It is amazing how soon this happens. Only the 8-family child 11700624 makes it on its own almost to the end. $N_{\text{max}}$ is the maximum value for the iteration process which occurs after $K_m$ iterations. The 3-family is childless.

References:

### Table 5. The first 180 iterations for N=11, N=1

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### Table 6. Some Wondrous Numbers and their Generators

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Note: This text is a placeholder for the actual content of the tables and figures.
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Table 6. Some Wondrous Numbers and their Genetic Relations, D=11
Table 7. The 7-family, Grandparent = 2038440

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Table 8. The B-family, Grandparent = 134680

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<th>N-one step before merging</th>
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<th>K</th>
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4. Iterating \( d(n) \) and \( o(n) \) - Two problems proposed by F. Smarandache

(a) Let \( n \) be a positive integer and \( d(n) \) the number of positive divisors of \( n \) including \( 1 \) and \( n \). Find the (smallest) \( k \) for which \( d(d(...d(n)...)) = 2 \) after \( k \) iterations, i.e. find \( k \) so that \( d_k(n) = 2 \).

\( d(n) \) is an important arithmetic function. We will look at its most important properties. The factors of \( p^n \), where \( p \) is a prime, are \( 1, p, p^2, ..., p^n \). Consequently \( d(p^n) = 1 + \alpha \).

The number of factors in \( n = p_1^{a_1}p_2^{a_2} \) is easily seen to be \( (1 + \alpha)(1 + \beta) \) from which the following important theorem follows:

If \( n_1 \) and \( n_2 \) have no common divisor, i.e. are relatively prime, \( n_1n_2 = 1 \), then \( d(n_1n_2) = d(n_1)d(n_2) \). We say that the order is multiplicative.

With \( n \) written in standard form \( n = p_1^{a_1}p_2^{a_2}...p_s^{a_s} \) we have

\[
d(n) = (1 + \alpha)(1 + \beta)...(1 + \gamma)
\]

We can now state:

\( d(n) = 2 \) if and only if \( n = 1 \).

\( d(n) = 2 \) if and only if \( n \) is a prime number.

From the above properties we see that \( d(n) \) is a measure of the number of factors of \( n \). The larger the number of factors of \( n \), the larger the value of \( d(n) \). This makes \( d(n) \) a useful tool in number theory.

Smarandache's question [1]: How many iterations \( k \) are required for \( d(n) \) to become \( 2 \)?

Before looking at this problem let's make an important observation:

Given an arbitrarily large positive integer \( n \), there are infinitely many integers \( n \) for which \( d_k(n) > 2 \).

Here is why: Let \( p_1, p_2, ..., p_s \) be odd primes (not necessarily distinct) and let \( n = p_1^{a_1}p_2^{a_2}...p_s^{a_s} \). Then

\[
d(n) = (1 + \alpha)(1 + \beta)...(1 + \gamma) = 2^{a_1}2^{a_2}...2^{a_s} = 2^k
\]

where \( k \) is the number of iterations required for \( d(n) \) to become \( 2 \). Thus, there are infinitely many integers \( n \) for which \( d_k(n) > 2 \).
If \( n_1 \) and \( n_2 \) have no common divisor, i.e. are relatively prime which we write \((n_1, n_2)=1\), then \( d(n_1n_2)=d(n_1)d(n_2)\). We say that the arithmetic function \( d(n) \) is multiplicative.

With \( n \) written in standard form \( n=p_1^{a_1}p_2^{a_2}...p_i^i \) we have

\[
d(n) = (1+a_1)(1+a_2)...(1+a_i)
\]  

(1)

We can now state:

\[
d(n)=n \text{ for all } n.
\]

\[
d(n)=1 \text{ if and only if } n=1.
\]

\[
d(n)=2 \text{ if and only if } n \text{ is a prime number.}
\]

From the above properties we see that \( d(n) \) is a measure of how far \( n \) is from being a prime number. The larger the number of factors of \( n \) the larger is \( d(n) \), \( d(n) \) being equal to 2 only when \( n \) is a prime number. This makes it interesting to try to answer Smarandache’s question [1]: How many iterations \( n \) are required in \( d_0(n)=2 \)?

Before looking at this problem let’s make an important observation:

Given an arbitrarily large positive integer \( k \) we can always construct infinitely many integers \( n \) for which \( d_0(n)=2 \) for all \( i<k \) and \( d_0(n)=2 \).

Here is how: Let \( p_1, p_2, ... p_k \) be odd primes (not necessarily distinct). Make the following series of constructions:

\[
n_1 = p_1
\]

\[
d(n_2) = 2
\]

\[
n_2 = p_2^{a_2-1}
\]

\[
d(n_2) = p_1
\]

\[
n_3 = p_3^{a_3-1} = p_3^{a_3-1}
\]

\[
d(n_3) = p_2^{a_2-1}
\]

\[
n_4 = p_4^{a_4-1}
\]

\[
d(n_4) = n_2
\]

\[
...
\]

\[
n_k = p_k^{a_k-1}
\]

\[
d(n_k) = n_{k-1}
\]

So that for \( n = n_k \) we have \( d_0(n) = 2 \) while \( d_0(n) > 2 \) for \( i < k \). Since we can choose our primes anyway we like as long as \( p_i \neq 2 \) this construction can be carried out in

53
infinitely many ways. If we ask for the smallest k for which \( d_k(n) = 2 \) for all n then the answer is that such a k does not exist.

Now to the problem. Factorizations and applications of (1) have been used to calculate k as a function of n for n<100, table 9. This does not tell us much about the general behaviour of \( d_k(n) \). Table 10 provides some interesting cumulative statistics for \( n \leq 10^6 \). No more than 6 iterations are required for any \( n \leq 10^6 \). It seems strange that the column for k = 7 remains empty for \( n \leq 10^5 \) and \( n \leq 10^6 \) in particular in view of the regular behaviour in the column for k = 3 and our previous observation that k can be arbitrarily large for properly chosen sufficiently large n. This calls for further study.

<table>
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<tr>
<th>n</th>
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<th>n</th>
<th>k</th>
<th>n</th>
<th>k</th>
<th>n</th>
<th>k</th>
<th>n</th>
<th>k</th>
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</table>

Let \( p# \) denote the product of all prime numbers less than or equal to p and consider the largest number r of distinct prime numbers which are needed to construct any integer \( \leq 10^6 \) i.e. \( p# \leq 10^6 < p_r# \). With these primes consider all possible constructions

\[
2^{3^\beta} \cdots p_r^2 \leq 10^6
\]  

(2)

This does not mean constructing all \( n \leq 10^6 \) but it does mean arriving at a structure into which all prime factorizations of \( n \leq 10^6 \) fits. This will be so because any number \( \leq 10^6 \) not produced by (2) will have fewer prime factors and smaller powers than one or more of the integers produced by (2). To illustrate this let's look at the case \( n \leq 100 \). We have 2\(3^5 \leq 100 < 2\times 3\times 5^2 \). We will therefore consider all possible construction \( 2^a\cdot 3^b\cdot 5^c \leq 100 \). These are obtained for \( a \leq 6, \beta \leq 4 \) and \( c \leq 2 \) resulting in table 11.

<table>
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<tr>
<th>n</th>
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<th>k=3</th>
<th>k=4</th>
</tr>
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<td>3</td>
<td>3</td>
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<tr>
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<td>7</td>
<td>34</td>
<td>28</td>
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<td>3181</td>
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Table 10. Number of iteration k required

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Table 11. All possible prime factorization constructions

Any number \( \leq 100 \) corresponds to one or more of the values listed in table 11 for \( n \leq 100 \). The above result for \( n \leq 10^6 \) is shown on table 10 to give a clear picture of the overall behaviour of \( d_k(n) \).

Finally we will be able to say something about \( d_k(n) \) for \( n \leq 10^{12} \). From table 10 it is seen that \( k \leq 6 \) for \( n \leq 100 \).
\[ y = 2 \text{ for all } n \text{ then} \]

It has been used to calculate some strange that the number of factors of \( n \) is equal in view of the observation that \( k \) can be 1 and further study.

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<th>( k=1 )</th>
<th>( k=2 )</th>
<th>( k=3 )</th>
<th>( k=4 )</th>
<th>( k=5 )</th>
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<th>( k=7 )</th>
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</table>

Table 10. Number of iteration \( k \) required to arrive at \( d_n(n) = 2 \)

<table>
<thead>
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<th>( C = 1 )</th>
<th>( d_n(n) = 2 )</th>
</tr>
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<td>6,12</td>
</tr>
<tr>
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<td>7,20</td>
</tr>
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<td>9,30</td>
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<td>10,40</td>
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</table>

Table 11. All possible prime factorization combinations \( C \) for \( n \leq 100 \)

Any number \( n \leq 100 \) corresponds to one or more of these structures, for example 77 = 11 \cdot 7 \). This means that \( d(n) \) can only assume values listed in table 11 for \( n \leq 100 \). The above scheme has been computer implemented for \( n \leq 10^{12} \). The result is shown on table 12 and figure 1, which gives a clear picture of the overall behaviour of \( d(n) \).

Finally we will be able to say something about \( d_n(n) \). For \( n \leq 10^{12} \) we have \( d(n) \leq 6720 \times 10^4 \). From table 10 it is seen that \( k \leq 6 \) for \( n \leq 10^4 \) and we therefore conclude that \( k \leq 7 \) for \( n \leq 10^{12} \).
Table 12. Largest value of \( d(n) \) for \( n \leq 10^4, k \leq 12 \)

<table>
<thead>
<tr>
<th>( k )</th>
<th>Largest ( d )</th>
<th>Number of ( d ) values</th>
<th>Number of combinations</th>
<th>Prime comb. for largest ( d )</th>
<th>Corresponding ( n )</th>
<th>( \log(d) )</th>
</tr>
</thead>
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<td>64211111100</td>
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<td>8.8128</td>
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</tbody>
</table>

(b) Let \( \sigma(n) = \sum_{d|n,d \neq 0} d \) and \( m \) a given positive integer

which \( \sigma(\ldots \sigma(2)\ldots) \geq m \) after \( k \) iterations, i.e. \( \sigma_k(2) \geq m \)

Clearly \( k \) is a function of \( m \). It is a stepwise increased in iterations have been used to illustrate this for the interval

However, a far more interesting function to study is the index \( m(k) \) is growing so rapidly that numerical results are not representative. A better picture of the behavior of \( \sigma(n) \) is obtained

\[
I(k) = \ln(m)/k
\]

This function is represented in diagram 3 for the interval, through an interesting minimum for small values of \( k \) then to remain downwards convex. Does it approach a horizon?

Finally, a few iteration results (\( k,m \)): (1,3), (2,4), (3,7),

\[
(100,297264850845695968647768973532548424660684)
\]

Diagram 1. Largest value of \( \ln(d(n)) \) for \( n \leq 10^4 \).

Diagram 2. \( k \) as a function of \( m \).
(b) Let \( \sigma(n) = \sum_{d|n, d \neq 0} \sigma(d) \) and \( m \) a given positive integer. Find the smallest \( k \) for which \( \sigma(\sigma(... \sigma(2) ...)) \geq m \) after \( k \) iterations, i.e. \( \sigma_k(2) \geq m \).

Clearly \( k \) is a function of \( m \). It is a stepwise increasing function. The first six iterations have been used to illustrate this for the interval \( 2 \leq m \leq 25 \) in diagram 2.

However, a far more interesting function to study is the inverse of \( k(m) \). This function \( m(k) \) is growing so rapidly that numerical results are difficult to interpret and represent. A better picture of the behavior of \( \sigma_k(2) \) is obtained by studying the function

\[
f(k) = \ln(m)/k
\]

This function is represented in diagram 3 for the interval \( 1 \leq k \leq 100 \). After going through an interesting minimum for small values of \( k \), the curve flattens out. It seems to remain downwards convex. Does it approach a horizontal asymptote?

Finally, a few iteration results \((k,m)\): (1,3), (2,4), (3,7), (4,8), (5,15), (6,24), (7,60), ... (100,297264850845695968647768973532548424660684303655482359755571200).

Diagram 2. \( k \) as a function of \( m \).
Chapter IV

Diophantine Equations

1. Some Thoughts on the Equation \(|x^p - y^q| = k\)

In his book *Only Problems, Not Solutions* Smarandache refers to #20 as a conjecture rather than a problem:

Let \( k \) be a non-zero integer. There are only a finite number of integers \( p, q, x, y \) such that \( |x^p - y^q| = k \) is satisfied, with each greater than 1, to the equation.

In this study we will only consider the equation for \( p \neq q \). By the form \(|x^p - y^q| = k\) we only need to consider cases where \( p > q \), since for \( p < q \) we can swap the roles of \( x \) and \( y \).

For a given set of parameters \((p, q, k)\) it would then be interesting to be able to determine the number of solutions \((x, y)\). However, if this were possible then many more cases would have been done. So Smarandache’s statement is likely more conjectural, with numerical evidence rather than analytical reasoning. It is mainly through numerical experimentation that we will study this equation. The parameters will be restricted to values where \( p \geq q \).

We will use most of the solutions that are known, and then develop the most effective method of finding solutions to this Diophantine equation. It is inspired by the method used in the previous chapter.

Our first solution will be the number of solutions where \( x < 100 \) and \( y < 100 \) can be churned out in any running computer.

Consider

\[
x = \sqrt[p]{y^q} \pm k
\]

For sufficiently large \( y \) only \( x = \lceil \sqrt[p]{y^q} \rceil \) or \( x = \lfloor \sqrt[p]{y^q} \rfloor \) will be solutions corresponding to

1 Smarandache adds that “For \( k = 1 \) this was conjectured by Cassels ([2], 1976).” (Gamma 2/1986)
Chapter IV

Diophantine Equations

1. Some Thoughts on the Equation \([y^q - x^q = k]\)

In his book *Only Problems, Not Solutions* Smarandache formulated unsolved problem \#20 as a conjecture rather than a problem:

Let \(k\) be a non-zero integer. There are only a finite number of solutions in integers \(p, q, x, y\), each greater than 1, to the equation \(x^p - y^q = k\).\(^1\)

In this study we will only consider the equation for \(p=q\). By writing the equation in the form \(x^p - y^q = k\) we only need to consider cases where \(p > q\).

For a given set of parameters \((p,q,k)\) it would then be desirable to list this finite number of solutions \((x,y)\). However, if this were possible it would probably already have been done. So Smarandache’s statement is likely to be based on statistical evidence rather than analytical reasoning. It is mainly from the statistical point of view we will study this equation. The parameters will be restricted to \(k \leq 200\) and \(p \leq 9\). As in most studies of this nature Ubasic provides the most effective computer language. All solutions where \(x < 100\) and \(y < 100\) can be churned out in a couple of seconds. To go any further we need a general approach to avoid running through meaningless search intervals. Consider

\[ x = p \sqrt[2]{y^q} \pm k \]

For sufficiently large \(y\) only \(x = \left\lfloor \sqrt[2]{y^q} \right\rfloor\) or \(x = \left\lfloor \sqrt[2]{y^q} \right\rfloor\) can produce solutions corresponding to

\(^1\) Smarandache adds that "For \(k=1\) this was conjectured by Cassels (1953) and proved by Tijdeman 1976." (Gamma 2/1986)
Chapter IV

Diophantine Equations

1. Some Thoughts on the Equation \[ y^3 - x^q = k \]

In his book *Only Problems, Not Solutions* Smarandache formulated unsolved problem #20 as a conjecture rather than a problem:

Let \( k \) be a non-zero integer. There are only a finite number of solutions in integers \( p, q, x, y \), each greater than 1, to the equation \( x^p - y^q = k \). \(^1\)

In this study we will only consider the equation for \( p \neq q \). By writing the equation in the form \( x^p - y^q = k \) we only need to consider cases where \( p > q \).

For a given set of parameters \((p,q,k)\) it would then be desirable to list this finite number of solutions \((x,y)\). However, if this were possible it would probably already have been done. So Smarandache's statement is likely to be based on statistical evidence rather than analytical reasoning. It is mainly from the statistical point of view we will study this equation. The parameters will be restricted to \( k \leq 200 \) and \( p \leq 9 \). As in most studies of this nature M. B. Basset provides the most effective computer language. All solutions where \( x < 100 \) and \( y < 100 \) can be churned out in a couple of seconds. To go any further we need a general approach to avoid running through meaningless search intervals. Consider

\[
x = p \sqrt[3]{y^q} \pm k
\]

For sufficiently large \( y \) only \( x = \lceil \sqrt[3]{y^q} \rceil \) or \( x = \lfloor \sqrt[3]{y^q} \rfloor \) can produce solutions corresponding to

\(^1\) Smarandache adds that "For \( k=1 \) this was conjectured by Cassels (1953) and proved by Tijdeman 1976." (Gamma 2/1986)
\[ k = \left\lceil \sqrt[3]{y^p} \right\rceil^p \cdot y^q \]  
(1a)

or

\[ k = y^q - \left\lceil \sqrt[3]{y^p} \right\rceil^p \]  
(1b)

It is easy to imagine from (1a) and (1b) that the number of solutions thin out rapidly as we increase y. Let’s illustrate this for \( p = 3 \) and \( q = 2 \) by looking at the number of squares \( s \) which fit into the interval between \( (n-1)^3 \) and \( (n+1)^3 \), i.e. integers \( s \) for which

\[ (n-1)^3 < s^2 < (n+1)^3 \]  
(2)

Let \( s_{\text{min}} \) be the smallest and \( s_{\text{max}} \) the largest \( s \) which satisfies (2). We can then calculate the ratio \( f \) between the number of squares between \( (n-1)^3 \) and \( (n+1)^3 \) and the difference between these cubes, i.e.

\[ f = \frac{s_{\text{max}} - s_{\text{min}}}{(n+1)^3 - (n-1)^3} \]

To have a solution we must have an \( s \) such that \( |s^2 - n| = k \). The smaller \( f \) is the smaller is the chance for this to happen for our very limited range for \( k \). Let \( s_1 \) be the largest square which is smaller than \( n^3 \) and \( s_2 \) be the smallest square which is larger than \( n^3 \) then \( s_1 \) and \( s_2 \) give an indication of the behaviour of \( |s^2 - n^3| \). These two ways of looking at the problem are displayed in Table 1 for a sequence of values of \( n \) which have been chosen so that \( n \) is neither square nor cube.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( s^2 - n^3 )</th>
<th>( n^2 - s^2 )</th>
<th>( f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>19</td>
<td>4</td>
<td>3.95 \times 10^{-2}</td>
</tr>
<tr>
<td>50</td>
<td>316</td>
<td>391</td>
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<tr>
<td>5 \times 10^3</td>
<td>14761</td>
<td>7600</td>
<td>4.40 \times 10^{-3}</td>
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<td>5 \times 10^4</td>
<td>430916</td>
<td>276991</td>
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</tr>
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<td>2875998881</td>
<td>4195097900</td>
<td>1.41 \times 10^{-7}</td>
</tr>
<tr>
<td>5 \times 10^7</td>
<td>11203852544</td>
<td>1156827231</td>
<td>4.47 \times 10^{-11}</td>
</tr>
<tr>
<td>5 \times 10^8</td>
<td>513527672836</td>
<td>193579108351</td>
<td>1.41 \times 10^{-13}</td>
</tr>
<tr>
<td>5 \times 10^9</td>
<td>1151976475001</td>
<td>21208703299996</td>
<td>4.47 \times 10^{-14}</td>
</tr>
</tbody>
</table>

Although the search for solutions was extended to \( y = 1000 \) for the following values of \( k \):

6, 14, 21, 29, 34, 42, 43, 50, 52, 58, 62, 66, 69, 70, 72, 74, 76, 110, 111, 114, 123, 125, 130, 133, 134, 146, 149, 150, 154, 170, 178, 182, 187, 189, 195.

No solutions were found for \((p, q) = (9, 5), (9, 6), (9, 7)\) and \((9, 8)\).

**Computer search:** Calculations were carried out in Ubasicsp9. Most solutions occur for small values of \( y \) for which we used the \( \text{FLOOR} \) and the \( \text{FLOOR} \) (\( \lfloor \cdot \rfloor \)) functions. The \text{ROUND} function was used to round to the nearest integer. Calculations are carried out with real numbers to 19 decimal places (in Ubasic language). To safeguard against “near integer” solutions.

---

**Title:** Teach yourself C++ in 21 days

**Subtitle:** Learning features

- Master the foundations of the language
- Learn the C++ syntax
- Gain proficiency with the standard library
- Develop robust and efficient programs
Table 2. Largest \( y \)

<table>
<thead>
<tr>
<th>( k )</th>
<th>( p )</th>
<th>( q )</th>
<th>( y )</th>
<th>( x )</th>
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<td>2</td>
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<td>2</td>
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<td>6</td>
<td>529</td>
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<td>4</td>
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<td>23</td>
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<tr>
<td>96</td>
<td>7</td>
<td>5</td>
<td>2</td>
<td>2</td>
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<tr>
<td>64</td>
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<td>2</td>
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<td>80</td>
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<td>6</td>
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<tr>
<td>113</td>
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<td>4</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

Although the search for solutions was extended to \( y = 10000 \) no solutions were found for the following values of \( k \):


No solutions were found for \((p,q)=(9,5), (9,6), (9,7)\) and \((9,8)\).

**Computer search:** Calculations were carried out in Ubasic for \( y \leq 10000 \), \( k \leq 200 \) and \( p \leq 9 \). Most solutions occur for small values of \( y \) for which we cannot use the CEIL \( \lceil \cdot \rceil \) and the FLOOR \( \lfloor \cdot \rfloor \) functions. The ROUND function has been used instead. Calculations are carried out with real numbers to 19 decimal places (POINT 9 in the Ubasic language). To safeguard against "near integer" solutions a "proposed" solution is
recalculated with integers in a subroutine. A simple version of the program is given below:

```
10 point 9
20 Y=1:
30 for R%=3 to 9
40 for S%=2 to R%
50 for K%=1 to 200
60 Y=2
70 while Y<Y1
80 if (Y^[R%]*K%)<1 then X1=(K%-Y^[R%]*K%)^[1/S%]:goto 100
90 X1=(Y^[R%]*K%)^[1/S%]
100 X=round(X1)
110 if abs(X-X1)<10^[(-30)] then gosub 210
120 endif
130 X1=(Y^[R%]*K%)^[1/S%]
140 X=round(X1)
150 if abs(X-X1)<10^[(-30)] then gosub 210
160 endif
170 inc Y
180 wend
190 next;next;next
200 end
210 if abs(Y^[R%]*K%)<>K% then goto 240
220 if X<1 then goto 240
230 print R%,S%,Y,X,K%
240 return
```

The number of solutions for each set of parameters is given in table 3. The largest value of y, which occurs in a solution for each parameter set (p,q) is given in table 2, which confirms the indications for the rarity of solutions for large y given in table 1.

The largest number of solutions (11) occur for k=17. Several of these are due to the fact that no distinction is made between (x^k)^2 and (x^k)^2. These solutions are displayed in table 4. A limited search (y≤100) was carried out for 10≤ps≤20, k≤200. Only two solutions with y=2 were found. These results are shown in table 5.

**Conclusion:** Smarandache’s conjecture is well supported by the numerical results obtained in this study. The number of solutions diminish rapidly with increasing y, p and q.

---

Table 4. Solutions for k=17

<table>
<thead>
<tr>
<th>Sol. #</th>
<th>p</th>
<th>q</th>
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</thead>
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</tr>
<tr>
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</table>

Table 5. Solutions for 10≤ps≤20

<table>
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<th>q</th>
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Table 4. Solutions for K=17

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<th>X</th>
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</table>

Table 5. Solutions for K=20

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<tr>
<td>5</td>
<td>5</td>
</tr>
</tbody>
</table>

The program is given in Table 2.

The numerical results of these are due to the solutions are displayed in Table 4.

Only two

XML DOM and how to extend your existing Java

are including:

AXAPI in Java

L into Java database

Servlets JSP and Swing

knowledge and access strategies that change the way you work.
Table 3a. The number of solutions to $|y-x|=k$

<table>
<thead>
<tr>
<th>k/q</th>
<th>2</th>
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Table 3b. The number of solutions to $|y-x|=k$

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Table 3c. The number of solutions to $l_\mathbf{x} - \mathbf{y} \cdot p = k$

| $k / q$ | 2 | 2 | 2 | 3 | 3 | 4 | 2 | 3 | 4 | 5 | 2 | 3 | 4 | 5 | 2 | 3 | 4 | 5 | 3 | 4 | 5 | 6 | 7 | 2 | 3 | 4 | Sum |
| 100    | 6 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | 8 |
| 101    | 1 | 1 | | | | | | | | | | | | | | | | | | | | | | | | 2 |
| 103    | 1 | | | | | | | | | | | | | | | | | | | | | | | | 1 |
| 104    | 3 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | | 7 |
| 105    | 1 | 2 | | 1 | | | | | | | | | | | | | | | | | | | | 5 |
| 106    | 2 | | | | | | | | | | | | | | | | | | | | | | | | 2 |
| 107    | 1 | | | | | | | | | | | | | | | | | | | | | | | | 1 |
| 108    | 1 | | | | | | | | | | | | | | | | | | | | | | | | 1 |
| 109    | 1 | | | | | | | | | | | | | | | | | | | | | | | | 2 |
| 112    | 2 | 1 | 1 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | 9 |
| 113    | 4 | | 1 | | 1 | 1 | | | | | | | | | | | | | | | | | 7 |
| 115    | 1 | | | | | | | | | | | | | | | | | | | | | | | | 2 |
| 116    | 3 | | | | | | | | | | | | | | | | | | | | | | | | 3 |
| 117    | 1 | | | | | | | | | | | | | | | | | | | | | | | | 2 |
| 118    | 1 | | 1 | | | | | | | | | | | | | | | | | | | | | | 3 |
| 119    | 1 | | | | | | | | | | | | | | | | | | | | | | | | 2 |
| 120    | | | | | | | | | | | | | | | | | | | | | | | | | 1 |
| 121    | 2 | | | | | | | | | | | | | | | | | | | | | | | | 2 |
| 122    | | | | | | | | | | | | | | | | | | | | | | | | | 1 |
| 124    | 2 | 1 | | | | | | | | | | | | | | | | | | | | | | | 3 |
| 126    | | | | | | | | | | | | | | | | | | | | | | | | | 1 |
| 127    | 1 | | 1 | | | | | | | | | | | | | | | | | | | | | | 3 |
| 128    | 2 | 1 | | 1 | 1 | | | | | | | | | | | | | | | | | | 6 |
| 129    | 1 | | 1 | | | | | | | | | | | | | | | | | | | | | | 3 |
| 131    | 1 | | | | | | | | | | | | | | | | | | | | | | | | 3 |
| 132    | 1 | | 1 | | | | | | | | | | | | | | | | | | | | | | 3 |
| 135    | 3 | 1 | 1 | 1 | | | | | | | | | | | | | | | | | | | | 6 |
| 136    | | | | | | | | | | | | | | | | | | | | | | | | | 1 |
| 137    | | | | | | | | | | | | | | | | | | | | | | | | | 1 |
| 138    | | | | | | | | | | | | | | | | | | | | | | | | | 1 |
| 139    | | | | | | | | | | | | | | | | | | | | | | | | | 1 |
| 140    | | | | | | | | | | | | | | | | | | | | | | | | | 1 |
| 141    | 1 | | | | | | | | | | | | | | | | | | | | | | | | 2 |
| 142    | | | | | | | | | | | | | | | | | | | | | | | | | 1 |
| 143    | 1 | | | | | | | | | | | | | | | | | | | | | | | | 2 |
| 144    | 2 | 1 | | 1 | | | | | | | | | | | | | | | | | | | | 4 |
| 145    | 2 | | 1 | | | | | | | | | | | | | | | | | | | | | | 3 |

Table 3d. The number of solutions to $l_\mathbf{x} - \mathbf{y} \cdot p = k$

| $k / q$ | 2 | 2 | 3 | 2 | 3 | 4 | 2 | 3 | 4 | 5 | 2 | 3 | 4 | 5 | 2 | 3 | 4 | 5 | 2 | 3 | 4 | Sum |
| 147    | 2 | | | | | | | | | | | | | | | | | | | | | | | | 2 |
| 148    | 1 | | | | | | | | | | | | | | | | | | | | | | | | 1 |
| 151    | | | | | | | | | | | | | | | | | | | | | | | | | 1 |
| 152    | 3 | | | | | | | | | | | | | | | | | | | | | | | | 3 |
| 153    | 1 | | | | | | | | | | | | | | | | | | | | | | | | 1 |
| 154    | | | | | | | | | | | | | | | | | | | | | | | | | 1 |
| 155    | | | | | | | | | | | | | | | | | | | | | | | | | 1 |
| 156    | | | | | | | | | | | | | | | | | | | | | | | | | 1 |
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| 159    | | | | | | | | | | | | | | | | | | | | | | | | | 1 |
| 161    | 2 | | | | | | | | | | | | | | | | | | | | | | | | 2 |
| 162    | | | | | | | | | | | | | | | | | | | | | | | | | 1 |
| 163    | 1 | | | | | | | | | | | | | | | | | | | | | | | | 1 |
| 165    | | | | | | | | | | | | | | | | | | | | | | | | | 1 |
| 166    | | | | | | | | | | | | | | | | | | | | | | | | | 1 |
| 168    | | | | | | | | | | | | | | | | | | | | | | | | | 1 |
| 169    | 2 | | | | | | | | | | | | | | | | | | | | | | | | 2 |
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| 174    | 1 | | | | | | | | | | | | | | | | | | | | | | | | 1 |
| 175    | 1 | 2 | | | | | | | | | | | | | | | | | | | | | | | 3 |
| 179    | | | | | | | | | | | | | | | | | | | | | | | | | 1 |
| 180    | 2 | | | | | | | | | | | | | | | | | | | | | | | | 1 |
| 193    | | | | | | | | | | | | | | | | | | | | | | | | | 1 |
| 194    | | | | | | | | | | | | | | | | | | | | | | | | | 1 |
| 195    | | | | | | | | | | | | | | | | | | | | | | | | | 1 |
| 196    | 1 | | | | | | | | | | | | | | | | | | | | | | | | 1 |
| 197    | 1 | | | | | | | | | | | | | | | | | | | | | | | | 1 |
| 198    | 2 | | | | | | | | | | | | | | | | | | | | | | | | 1 |
| 199    | 2 | | | | | | | | | | | | | | | | | | | | | | | | 1 |
| 200    | 3 | 1 | | | | | | | | | | | | | | | | | | | | | | | 4 |
## Table 3d. The number of solutions to $|y - x| = k$

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$\text{Sum}$: 2

## CONCEPTS AND APPLICATIONS

XML changes the way you...extend your existing Java knowledge with Servlets, JSP, and Swing. a...access strategies that...transformation using XSLT...the XML DOM and how to...XML into Java database...e SAX API in Java...software including:

- Server

Samples from the book
2. The Equation \(7(p^4+q^4+r^4+s^4+t^4) - 5(p^2+q^2+r^2+s^2+t^2)^2 + 90pqrs = 0\)

This equation first appeared in the Numbers Count column of the Personal Computer World in March 1986. At that time the author found two until then unknown solutions \((p,q,r,s,t)=(87,42,6,3,3)\) and \((p,q,r,s,t)=(99,97,39,13,2)\). This took approximately 75 hrs on an 8086 processor running a 4.7 Mhz. Now, in 1996, these results were reproduced using the same program in 21 minutes on a Pentium 100 Mhz. The method used is reproduced below together with a number of new solutions.

Consider the function

\[F(p,q,r,s,t) = 7(p^4+q^4+r^4+s^4+t^4) - 5(p^2+q^2+r^2+s^2+t^2)^2 + 90pqrs\]

where

\[G(x) = 28x^3 + 22x^2 + 8x + 2\]

\[H(x) = 2x + 1\]

\[C = p^2+q^2+r^2+s^2+t^2\] with one of the constants replaced by \(x\)

\[D = 90pqrs\] with the same constant as above replaced by \(x\)

This permits us to calculate \(F(x+\ldots+1)\), \(G(x), H(x), C\) and \(D\). When, in the next step, one of the unknowns \(p, q, r, s, t\) is increased by one the following replacements must be made

\[C = C + H(x)\]

\[D = D(x+1)\]

Without imposing any restrictions we can assume \(p \geq q \geq r \geq s \geq t\). For a given value of \(p\) the search will be conducted for descending values of the other unknowns. For given values of \(p, q, r, s, t\) consider the function

\[g(t) = 7(e^t^2) - 5(a + 1)^2 + bt\]

where \(e = p^2+q^2+r^2+s^2\), \(a = p^2+q^2+r^2+s^2\) and \(b = 90pqrs\)

We have

\[g'(t) = 8t^2 - 20at + b\]

\[g''(t) = 24t^2 - 20a\]

\[g''(t) = 0\] for \(t_n = \sqrt{\frac{20a}{24}}\) which gives \(g_{\max} = 1\).

Case 1: \(g_{\max} > 0\). Since \(g'(t) > 0\) it follows that \(g(t)\) is an increasing function for \(t > 0\). If we have found \(t_n\) such that \(t > t_n\) and the search can be interrupted for \(t = t_n\).

Case 2: \(g_{\max} \leq 0\). For \(t > t_n\) the function \(g(t)\) is convex and where the function is positive and increasing. The search

The above method has been used in a computer program implementation on Pentium 100 Mhz computer has revealed a complete list of all solutions for \(p \leq 400\) is given in Table 4.

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With

XML Development

Michael C. Daconta is president of Michael C. Daconta, Inc., a consulting firm specializing in the development of Java and XML applications. Understand the XML process with the help of XML development with Java. This text provides a comprehensive guide to integrating XML into your Java programs.

We have $g(0) = 8 \cdot 10^2$. We can convert the function $g(t)$ into an increasing function.

The above method has been used in a computer program written in C++. The complete list of all solutions for $g(0)$ is given in Table 4.
3. The Equation \( y = 2 \cdot x_1 x_2 \ldots x_k + 1 \)

Conjecture:

Let \( k \geq 2 \) be a positive integer. The diophantine equation \( y = 2 \cdot x_1 x_2 \ldots x_k + 1 \) has infinitely many solutions in distinct primes \( x_1, x_2, \ldots, x_k \).

This is an unsolved problem number 11 in Smarandache’s book "Only Problems. Not Solutions." The word "distinct" has been added by the author. The purpose of this study is to see how stable this conjecture is. This is done through a computer analysis of all possible solutions for \( y \leq 10^6 \). (A very thin layer when surfing the ocean of integers but big enough to take a bit of time on the computer when it comes to calculations in fact more than 100 hrs). From the computational point of view there is no reason to exclude \( k=1 \). The interval \( 0 < y < 10^6 \) is divided into 10 sub-intervals of equal length:

- Interval #1: \( 0 < y < 10^6 \)
- Interval #2: \( 10^6 < y < 2 \cdot 10^6 \)
- Interval #3: \( 2 \cdot 10^6 < y < 3 \cdot 10^6 \)
- Interval #10: \( 9 \cdot 10^6 < y < 10^7 \)

The endpoints are excluded since these do not contribute to the number of solutions.

Consider

\[ t = (y-1)/2 = x_1 x_2 \ldots x_k \]  \( (1) \)

The task is to identify sequentially all square free numbers \( n < 10^6 \). For each square free number with \( k \) distinct prime factors we calculate the corresponding number \( y \) and test whether it is prime or not. The number of primes is denoted \( m_k \) and the number of square free numbers is denoted \( n_k \). \( m_k \) and \( n_k \) are recorded for each interval and the frequency of solutions \( f_k = m_k/n_k \) is calculated. The result is shown in table 5.

The same result is shown in diagram 1, which conveys a good visualization of a result obtained through surfing on a small area of the ocean of integers.

Let’s make a few observations:

Why the irregularities for \( k=8 \) and \( k=9 \)? The smallest square free integer with \( k \) prime factors is \( p_k \# \) where \( p_k \) is the \( k \)th prime. For \( k=8 \) and \( k=9 \) this means there can be no solutions for \( y < 2 \cdot 19 \# + 1 = 19399381 \) and \( y < 2 \cdot 23 \# + 1 = 446185741 \) respectively. The samples for \( k=8 \) and \( k=9 \) are therefore too small to give a true picture - randomness takes precedence over mass behaviour.
Table 5. Frequency of solutions

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<td>0.1036</td>
<td>0.1244</td>
<td>0.1508</td>
<td>0.1883</td>
<td>0.2359</td>
<td>0.6667</td>
</tr>
<tr>
<td>10</td>
<td>0.0639</td>
<td>0.0738</td>
<td>0.0867</td>
<td>0.1030</td>
<td>0.1238</td>
<td>0.1498</td>
<td>0.1846</td>
<td>0.2120</td>
<td>0.4000</td>
</tr>
</tbody>
</table>

Diagram 1. Frequency of solutions
With the previous observation in mind let's compare the frequencies for the first and the last interval. Diagram 2 shows $F_k$ for the intervals $0<y<10^8$ and $9 \cdot 10^8<y<10^9$.

![Diagram 2](image)

**Observation:** When not too close to $p_i$, the frequency smoothly increases with increasing $k$. This is a good support for Samrandache's conjecture.

Let's now take a closer look at how the frequency of solutions behave for distinct values of $k$. Diagram 3 shows that the frequency decreases slightly for all $k$ as larger integers are included. Is there an asymptote for each $k$? The frequency of solutions increases as we increase $k$. It should be noted that it is the ratio between the number of solutions and the number of square free numbers in an interval determined by $t$ in (1) which depicted. This is of course different from the number of square free numbers in the interval for $y$ because when $y$ runs through the interval $a<y<b$ then we consider square free numbers $s$ which obey

$$(a-1/2)^2<s<(b-1/2)$$

As a by-product to this study we have information on how many squares are associated with 1, 2, ..., 9 prime factors there are in our different intervals. In Table 6, where, as we have seen, we now only have five intervals.

| Interval #    | $0<y<10^8$ | $10^8<y<2 \cdot 10^8$ | $2 \cdot 10^8<y<3 \cdot 10^8$ | $3 \cdot 10^8<y<4 \cdot 10^8$ | $4 \cdot 10^8<y<5 \cdot 10^8$
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Interval #1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Interval #2</td>
<td>9</td>
<td>15</td>
<td>20</td>
<td>25</td>
<td>30</td>
</tr>
<tr>
<td>Interval #3</td>
<td>30</td>
<td>45</td>
<td>60</td>
<td>75</td>
<td>90</td>
</tr>
<tr>
<td>Interval #4</td>
<td>60</td>
<td>75</td>
<td>90</td>
<td>105</td>
<td>120</td>
</tr>
<tr>
<td>Interval #5</td>
<td>90</td>
<td>105</td>
<td>120</td>
<td>135</td>
<td>150</td>
</tr>
</tbody>
</table>

**Diagram 3.**

<table>
<thead>
<tr>
<th>Table 6. Number of square free numbers</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
</tr>
<tr>
<td>-------------</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
</tbody>
</table>

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As a bi-product to this study we have information on how many square free integers with 1, 2, ..., 9 prime factors there are in our different intervals. This is shown in Table 6, where, as we have seen, we now only have five intervals:

<table>
<thead>
<tr>
<th>Interval #</th>
<th>(10^k \leq x &lt; 10^{k+1})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interval 1</td>
<td>(k=10^0)</td>
</tr>
<tr>
<td>Interval 2</td>
<td>(10^0 \leq x &lt; 2 \cdot 10^0)</td>
</tr>
<tr>
<td>Interval 3</td>
<td>(2 \cdot 10^0 \leq x &lt; 3 \cdot 10^0)</td>
</tr>
<tr>
<td>Interval 4</td>
<td>(3 \cdot 10^0 \leq x &lt; 4 \cdot 10^0)</td>
</tr>
<tr>
<td>Interval 5</td>
<td>(4 \cdot 10^0 \leq x &lt; 5 \cdot 10^0)</td>
</tr>
</tbody>
</table>

Diagonal 3.

Table 6. Number of square free numbers

<table>
<thead>
<tr>
<th>#</th>
<th>(k=1)</th>
<th>(k=2)</th>
<th>(k=3)</th>
<th>(k=4)</th>
<th>(k=5)</th>
<th>(k=6)</th>
<th>(k=7)</th>
<th>(k=8)</th>
<th>(k=9)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5761455</td>
<td>17426029</td>
<td>20710806</td>
<td>12364826</td>
<td>3884936</td>
<td>605939</td>
<td>38186</td>
<td>516</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>5317482</td>
<td>16560025</td>
<td>20599998</td>
<td>13029165</td>
<td>4476048</td>
<td>799963</td>
<td>63462</td>
<td>1409</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>5173395</td>
<td>16270874</td>
<td>20457818</td>
<td>13243252</td>
<td>4689541</td>
<td>879765</td>
<td>76114</td>
<td>2060</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>5064001</td>
<td>16085983</td>
<td>20402004</td>
<td>1337430</td>
<td>4825914</td>
<td>932513</td>
<td>84968</td>
<td>2560</td>
<td>6</td>
</tr>
<tr>
<td>5</td>
<td>5019541</td>
<td>15951738</td>
<td>20359052</td>
<td>13468885</td>
<td>4926227</td>
<td>972398</td>
<td>91767</td>
<td>3005</td>
<td>8</td>
</tr>
</tbody>
</table>

CONCEPTS AND APPLICATIONS

XML changes the way you extend your existing Java knowledge with Servlets, JSP, and Swing access strategies that transformation using XSLT the XML DOM and how to use SAX API in Java.

Software including:

ic Server

Samples from the book
Diagram 4 shows a cumulative representation of the number of square free numbers with $k=1, 2, 3, ..., 9$ prime factors. A square free number with only one prime factor is indeed a prime number so the first column in table 6 shows the number of prime numbers in the interval.

The illustration on the cover is another version of the above diagram for $n<5 \cdot 10^7$.

Finally, a problem often leads to other problems. Let's go back to iterations. In the introduction to chapter III it was said that iterations result in an invariant, a loop or divergence. Is there really no other possibility? Let's look at this:

We define $u_{k+1} = 2u_k + 1$ where $u_1$ is a prime number. $u_{k+1}$ will have the property of being a prime number or not being a prime number.

In this case we can give an explicit formula for $u_{k+1}$ in terms of $u_k$. One easily finds $u_{k+1} = 2(u_k + 1) - 1$.

How do we characterize the result of iterations in relation to something we are interested? Indefinite?

How many times can we iterate $u_{k+1} = 2u_k + 1$ preserving prime? It is eight times resulting in the following series of nine primes:

305192579, 610385159, 1220770319, 2441540639, 4883081279, 9766162559, 19532325119, 39064630239, 78128740479

Which is the first series with 10 terms - or 11? Maybe we no
How do we characterize the result of iterations in relation to the property in which we are interested? Indefinite?

How many times can we iterate \( u_{k+1} = 2 \cdot u_k + 1 \) preserving primality? For \( u_1 = 305192579 \), it is eight times resulting in the following series of nine primes:

\[
305192579, 610385159, 1220770319, 2441540639, 4883081279, 9766162559, 19532325119, 39064650239, 78129300479
\]

Which is the first series with 10 terms - or 11? Maybe we need some deep sea diving!
The Smarandache Cell Function

**Definition:** For a positive integer $n$ the Smarandache cell function of order $k$ is defined through

$$S_k(n) = m$$

where $m$ is the smallest positive integer for which $n$ divides $m^k$.

In the study of this function we will make frequent use of the ceil function defined as follows:

$$\lceil x \rceil = \text{the smallest integer not less than } x.$$

The following properties follow directly from the above definitions:

1. $S_1(n) = n$
2. $S_k(p^a) = p^\lceil a/k \rceil$ for any prime number $p$.
3. For distinct primes $p, q, \ldots r$ we have $S_k(p^a q^b \ldots r^c) = p^\lceil a/k \rceil q^\lceil b/k \rceil \ldots r^\lceil c/k \rceil$.

Henry Ibstedt

$\ 9.95$